ECE 480
Winter 2000
Lecture Notes

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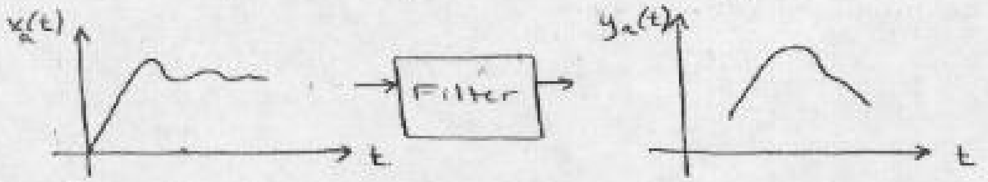
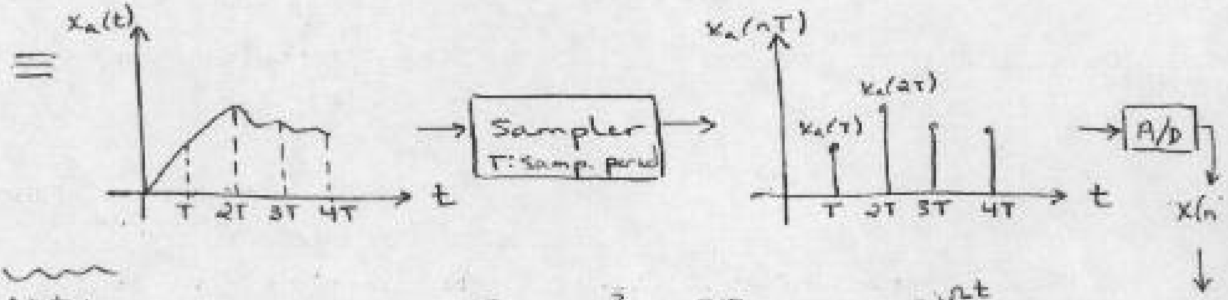
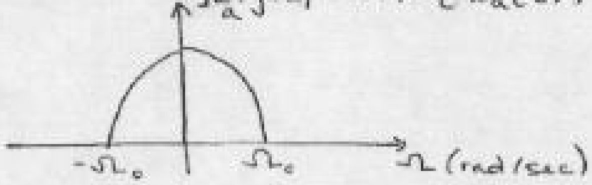
Intro to Digital Signal Processing





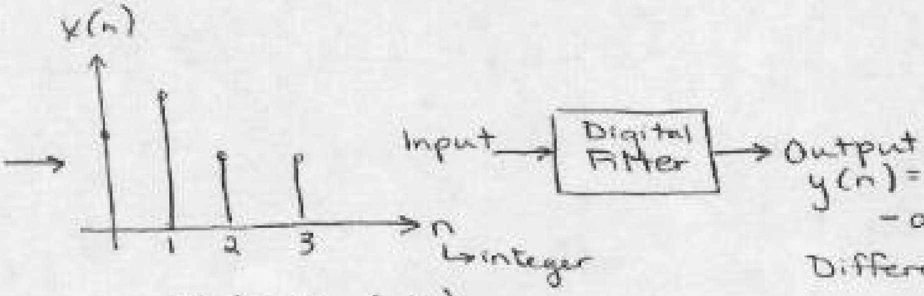
Note:

$X_a(j\Omega) = \text{F.T.}\{x_a(t)\} = \int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t}\,dt$



$X_a(j\Omega) = 0$ for $\Omega > |\Omega_0|$

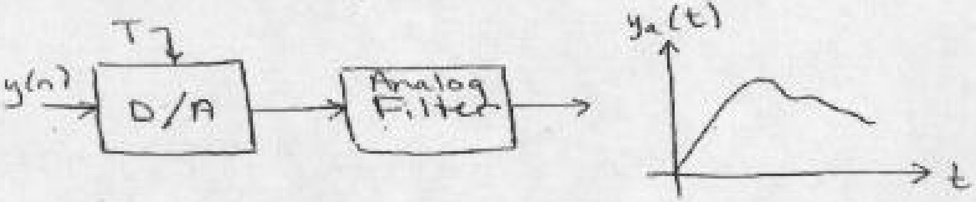
$\underset{\hookrightarrow \text{rad/sec}}{\Omega_0} = 2\pi \underset{\hookrightarrow \text{Hz}}{f_0}$ $\quad$ $f_{samp} = \frac{1}{T} > 2f_0$



$y(n) = b_0 x(n) + b_1 x(n-1$
$- a_1 y(n-1) - a_2 y(n-$

Difference Equation

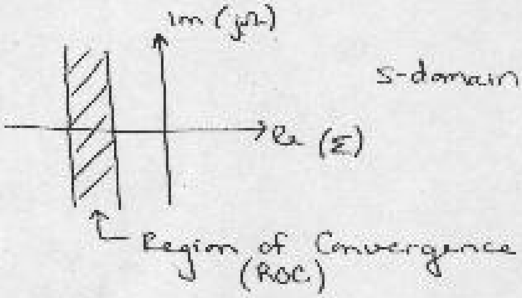
$x(n) = x_a(nT)$



Z-transform

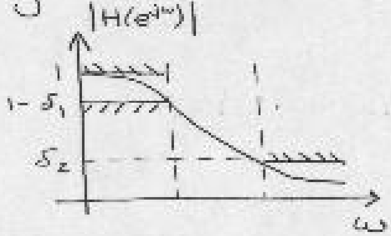
Laplace Transform:

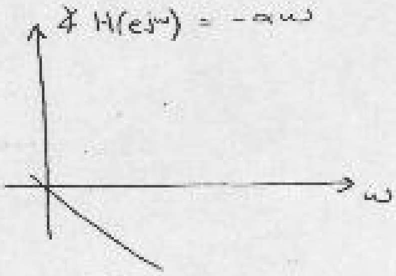
$$x_a(t) \xleftrightarrow{L.T.} X_a(s) = \int_0^\infty x_a(t) e^{-st} dt$$



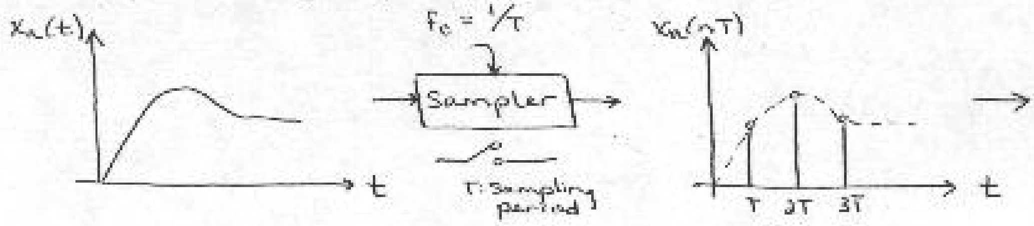
Z-transform applied to Difference Equ.
to convert to Algebraic Equ.

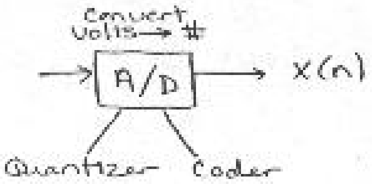
Digital Filters

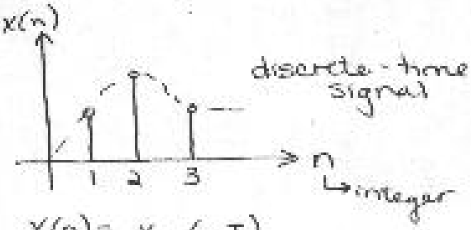




Discrete Time Signals





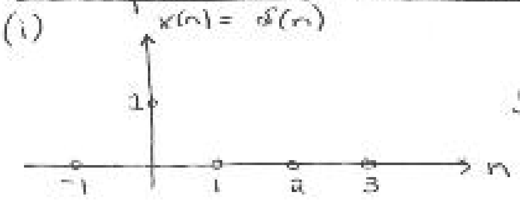


$$x(n) = x_a(nT)$$

$$n = 0, \pm 1, \pm 2, \pm 3, \cdots$$

Examples of Discrete-Time Signals (Sequences)

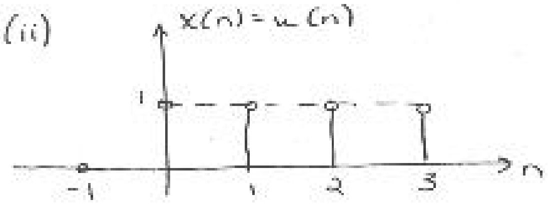
(i)



Unit Sample Sequence

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

(ii)



Unit Step Sequence

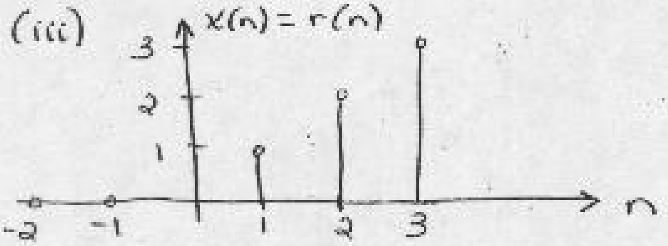
$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$\delta(n) = u(n) - u(n-1)$ First Difference

$\delta(n) = u(n) - u(n-1)$ First Difference

$$u(n) = \sum_{k=0}^{\infty} \delta(n-k)$$

$$= \sum_{k=-\infty}^{n} \delta(k)$$

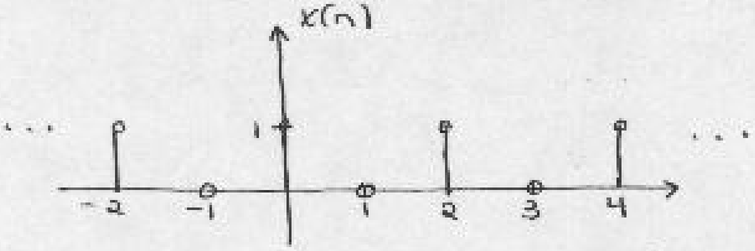


Unit Ramp Sequence

$$r(n) = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$r(n) = n u(n)$$

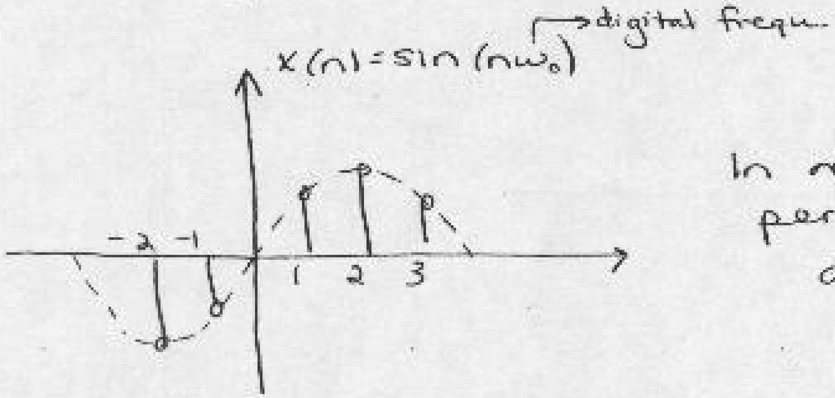
Periodic Sequences



$x(n) = x(n+2) = x(n-$

$= x(n+2k)$

$k = 0, \pm 1, \pm 2$



In order to be periodic, need to define ω_0

Periodic Sequences

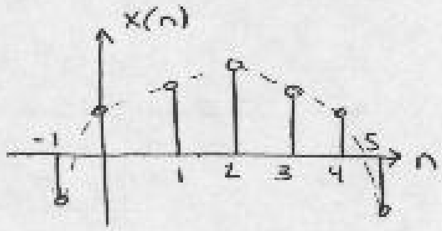
$x(n) = x(n+N) = x(n+2N) = \cdots = x(n+kN)$

$k = 0, \pm 1, \pm 2, \cdots$

N = period (samples)

ex:

$x(n) = \cos(\omega_0 n + \phi)$



Periodic? If yes, find the period N

Assume that $x(n)$ is periodic with period N

$x(n) = x(n+N) = x(n+kN)$

$\therefore \cos(\omega n + \phi) = \cos(\omega(n+N) + \phi)$

$= \cos(\omega n + \phi + \omega N)$

$\Rightarrow \omega N = \overset{\uparrow \text{integer}}{k} 2\pi$

$\frac{\omega}{2\pi} = \frac{k}{N}$ rational number relatively prime

If $\frac{\omega}{2\pi}$ is a rational number $\Rightarrow$ sequence is periodic, otherwise it is not.

(ω is the digital frequency)

ex:

1) $x(n) = \sin\left(\frac{3\pi}{4} n + \phi\right)$

$\omega = \frac{3\pi}{4}$ radians

Test: $\frac{\omega}{2\pi} = \frac{3\pi/4}{2\pi} = \frac{3}{8} \rightarrow N = 8$ Periodic

$\sin\left(\frac{3\pi}{4}(n+8) + \phi\right) = \sin\left(\frac{3\pi}{4} n + \phi + 6\pi\right)$

$= \sin\left(\frac{3\pi}{4} n + \phi\right)$ ✓

Ex:

$$x(n) = \cos\left(\frac{\sqrt{\pi}}{2} n + \phi\right)$$

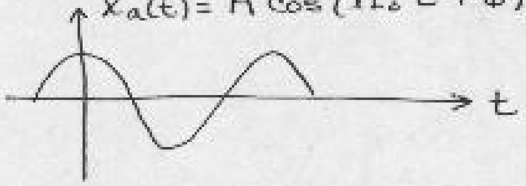
$\omega = \frac{\sqrt{\pi}}{2}$ radians

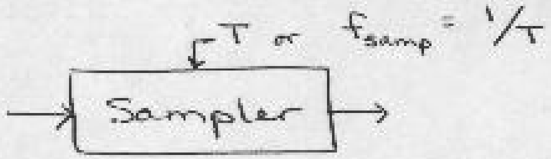
Test: $\frac{\omega}{2\pi} = \frac{\sqrt{\pi}/2}{2\pi}$ irrational number

Not periodic

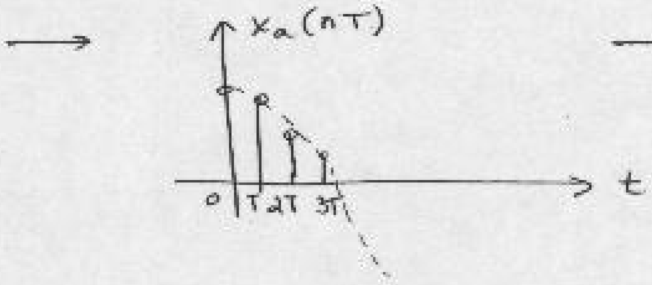
Sampling of Continuous-time Sine Waves

$x_a(t) = A\cos(\Omega_0 t + \phi)$

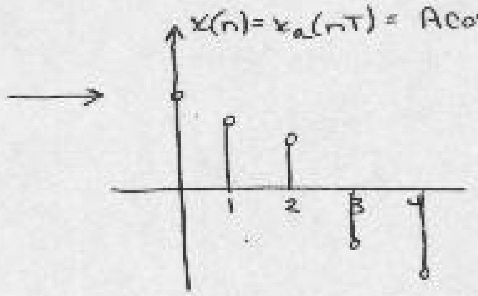




(Ω_0 = freq in rad/sec)



$x(n) = x_a(nT) = A\cos(nT\Omega_0 + \phi) = A\cos(\Omega_0 T n + \phi) = A\cos(\omega_0 n + \phi)$



$\omega_0 = \Omega_0 T$

ω_0 = digital freq.

Ω_0 = analog freq

T = Sampling period

$T = 1/f_{samp}$

$$\omega = \Omega T = \frac{\Omega}{f_{samp}} = \frac{2\pi f}{f_{samp}}$$

Analog Freq.		Digital
f, Hz	Ω, rad/sec	ω, rad
0	0	0
$0.1 f_{samp}$	$0.2\pi f_{samp}$	0.2π
$0.2 f_{samp}$	$0.4\pi f_{samp}$	0.4π
$0.5 f_{samp}$	πf_{samp}	π
f_{samp}		2π
$1.1 f_{samp}$		2.2π

Cannot be distinguished

$$\omega_1 \longrightarrow \omega_1 + 2\pi$$

$$x_1(n) = \cos(\omega_1 n + \phi)$$

$$x_2(n) = \cos((\omega_1 + 2\pi)n + \phi)$$

$$x_1(n) \overset{?}{=} x_2(n)$$

Proof:

$$\begin{aligned} x_2(n) &= \cos((\omega_1 + 2\pi)n + \phi) \\ &= \cos(\omega_1 n + \phi + 2\pi n) \\ &= \cos(\omega_1 n + \phi) = x_1(n) \end{aligned}$$

$$\therefore \quad x_1(n) = x_2(n)$$

Can not distinguish between sequences that have $2\pi k$ added to them:

$$\omega_1 \longrightarrow \omega_1 + k2\pi$$

ex:

$f_{sampling} = 10$ kHz

$$f_1 = 1 \text{ kHz} \longrightarrow \omega_1 = \frac{2\pi f_1}{f_{samp}} = 0.2\pi \text{ rad.}$$

$$f_2 = 11 \text{ kHz} \longrightarrow \omega_2 = 2.2\pi \equiv 0.2\pi$$

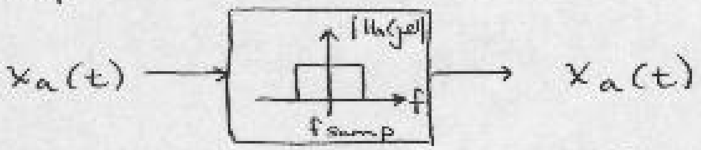
$$f_3 = 21 \text{ kHz} \longrightarrow \omega_3 \equiv 0.2\pi$$

$f_3 = 21$ kHz $\longrightarrow \omega_3 = 0.2\pi$

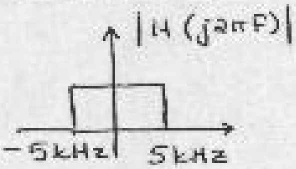
In order to prevent aliasing, need to

ex:

$f_{samp} = 10$ kHz



Anti-Aliasing LPF



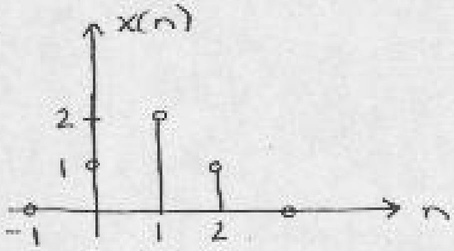
$f_c = f_{samp}/2$ allowed to pass

<u>Definition:</u>

The energy of a sequence, denoted by $\mathcal{E}$, is given by:

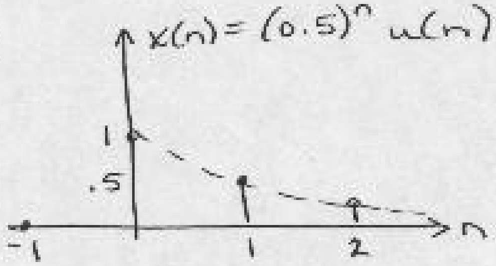
$$\mathcal{E} = \sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty \quad \text{(finite)}$$

ex:



$\mathcal{E} = 1^2 + 2^2 + 1^2 = 6$ finite

ex:



Infinite duration sequence

$$\mathcal{E} = \sum_{0}^{\infty} |x(n)|^2 = \sum_{0}^{\infty} (0.5)^{2n} = \sum_{0}^{\infty} (0.25)^n$$

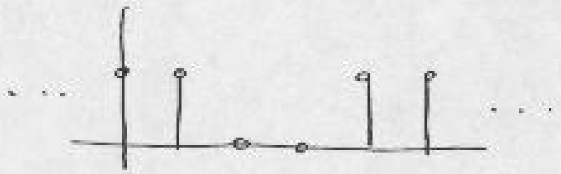
$$\mathcal{E} = \sum_0^\infty |x(n)|^2 = \sum_0^\infty (0.5)^{2n} = \sum_0^\infty (0.25)^n$$

Note: $S = k^0 + k^1 + k^2 + \cdots + k^n$

Geometric Series

$S = \underline{1-k^{n+1}} \rightarrow \lim S_n = \underline{\quad 1 \quad}$

$$\mathcal{E} = \sum_0^\infty \underbrace{(0.25)^n}_{k} = \frac{1}{1-k} = \frac{1}{0.75}$$



Periodic Sequence

$$P_{ave} = \frac{\mathcal{E}_{\text{one per}}}{N} = \text{constant}$$

$E(x) = 0$

$\sigma^2 = E((x-\bar{x})^2)$

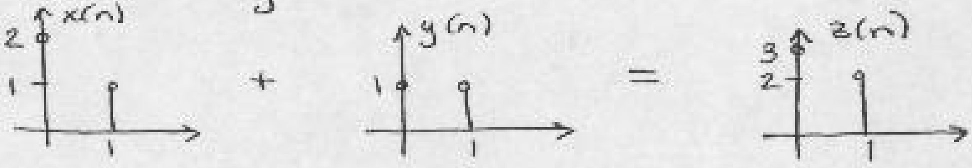
↳ Variance

σ = Standard deviation

Basic Operations

1) Addition

$z(n) = x(n) + y(n)$



2) Subtraction

$g(n) = x(n) - y(n)$ Special Case: $g(n) = x(n) - 1$

3) multiplication

$z(n) = x(n)y(n)$ Special Case: $g(n) = \underset{\uparrow \text{constant}}{\alpha} x(n)$

$e(n) = x(n)y(n)$ Special Case: $g(n) = \alpha x(n)$
↑ constant

4) Delay

Discrete-time Systems

Linear System

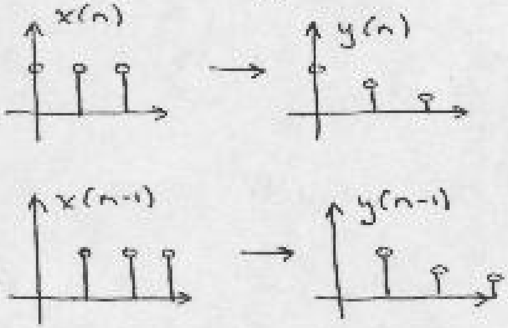
$x_1(n) \longrightarrow y_1(n)$
$x_2(n) \longrightarrow y_2(n)$
$\alpha x_1(n) + \beta x_2(n) \longrightarrow \alpha y_1(n) + \beta y_2(n)$

Superposition Principle

If $x(n)=0 \Rightarrow y(n)=0$

Time-Invariant

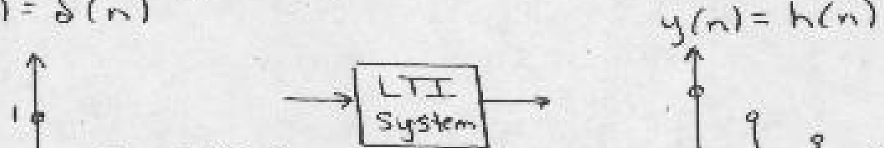
$x(n) \longrightarrow y(n)$
$x(n-n_0) \longrightarrow y(n-n_0)$

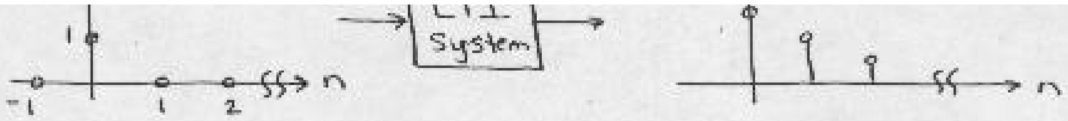


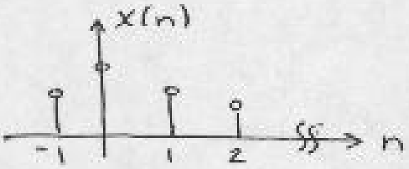
How to characterize the system?

1) Impulse Response, $h(n)$

$x(n) = \delta(n)$

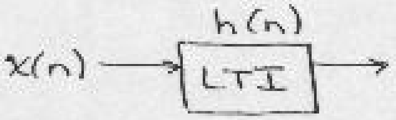






$$x(n) = \cdots + x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1) + \cdots$$

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$



$$y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) + x(-1)h(n+1) + \cdots$$

$$= \sum_{k=-1}^{\infty} x(k)h(n-k)$$

$$= x(n) * h(n) = h(n) * x(n)$$

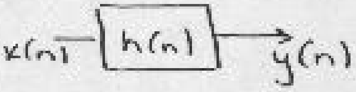
$$= \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

Some Basic Results

1)

2 systems in series (cascade)

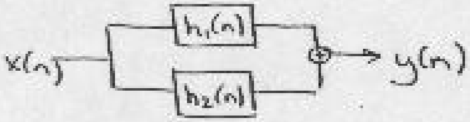
=



equivalent system

$$h(n) = h_1(n) * h_2(n)$$

2)



2 systems in parallel

=

equivalent system

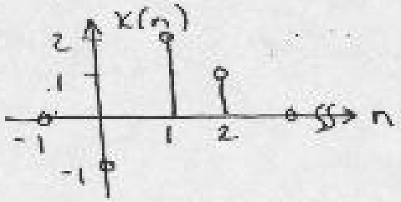
$$y(n) = x(n) * (h_1(n) + h_2(n))$$

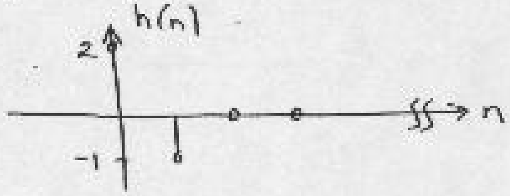
$$h(n) = h_1(n) + h_2(n)$$

Example

Find the output y(n) for a system described

Find the output y(n) for a system described by h(n) if the input is x(n)





Solution: $y(n) = x(n) * h(n) = h(n) * x(n)$

$$= \sum_{k=0}^{1} h(k)x(n-k)$$

$$y(0) = \sum_{k=0}^{1} h(k)x(0-k) = h(0)x(0) + h(1)x(-1) = -2$$

$$y(1) = \sum^{1} h(k)x(1-k) = h(0)x(1) + h(1)x(0) = 5$$

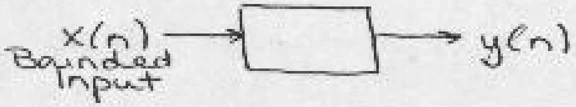
$$y(2) = \sum_{k=0}^{1} h(k)x(2-k) = 0$$

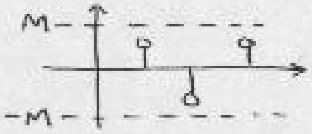
$$y(3) = -1$$

$$y(4) = y(5) = \cdots = 0$$

Important Concepts

1) Stability





$|x(n)| < M$ for all n $\Rightarrow$ $|y(n)| < M$ for all n

Stable if a bounded input produces a bounded output (BIBO Stable)

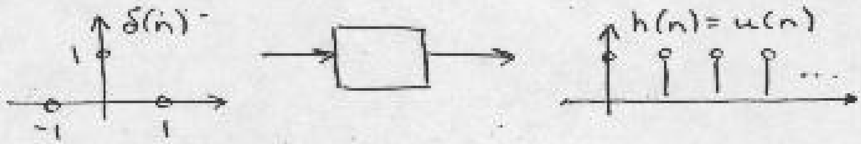
For a LTI system:

For a LTI system:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \text{ (finite)}$$

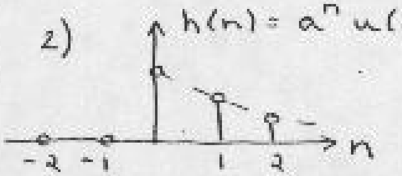
Examples:

1)



Unstable because $\sum_{0}^{\infty} u(n) = \infty$

2)



Stable

$\lim \frac{1-a^n}{} = \frac{1}{}$ ∀|a|<

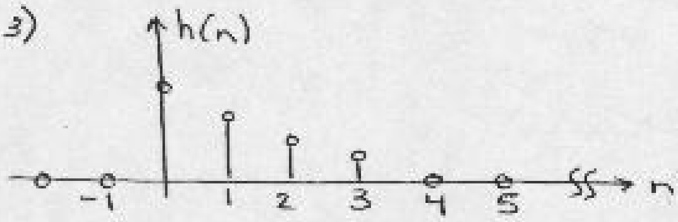
If $|a| \geq 1$

$\Rightarrow \sum |h(n)| = \infty$

Unstable

IIR (Infinite Impulse Response)

3)



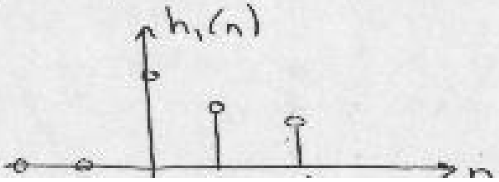
FIR (Finite Impulse Response)

Always Stable because

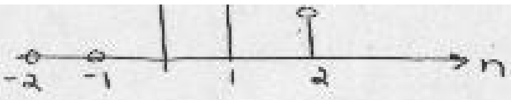
$\sum |h(n)|$ is always finite

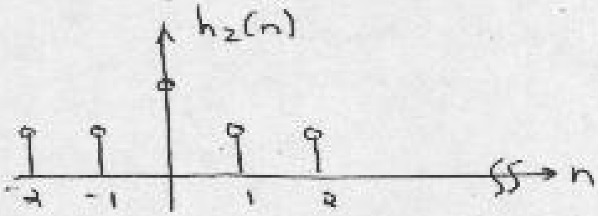
2) Causality

Examples



Causal





Non-Causal

A system is causal if + only if
$h(n) = 0 \ \forall \ n < 0$

Describing Discrete-time Systems in terms of Constant coefficient difference equations

Analog Systems:

$$\frac{d^2 y_a(t)}{dt^2} + \frac{a_1 dy_a(t)}{dt} + a_2 y_a(t)$$

$$= \frac{b_0 d^2 x_a(t)}{dt^2} + \frac{b_1 dx_a(t)}{dt} + b_2 x_a(t)$$

Discrete Time Systems:

$$\underset{\substack{\downarrow \\ =1}}{a_0} y(n) + a_1 y(n-1) + a_2 y(n-2) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2$$

Difference Equation

$y(n) = 0$ for $n < 0$

$y(n) = 0$ for $n<0$

$$\underset{\substack{\downarrow \\ \text{current} \\ \text{or present} \\ \text{output}}}{y(n)} = \underbrace{-a_1 y(n-1) - a_2 y(n-2)}_{\substack{\text{past} \\ \text{outputs}}} + \underbrace{b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)}_{\text{past and present inputs}}$$

Example:

$y(n) = 0.5y(n-1) + x(n)$

$y(0) = 0.5y(-1) + x(0) = x(0)$ because $y(n)=0$ for $n<0$

$y(1) = 0.5y(0) + x(1) = 0.5x(0) + x(1)$

$y(2) = \cdots$

$y(3) = \cdots$

If $x(n) = \delta(n) \Rightarrow y(n) = h(n)$

$h(n) = 0.5h(n-1) + \delta(n)$

$h(0) = 0.5h(-1) + \delta(0) = 1$

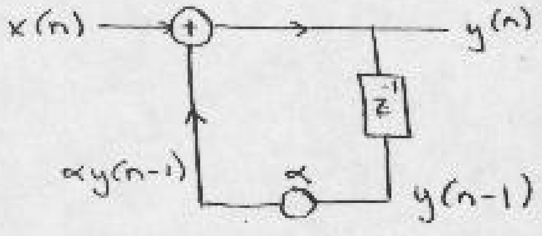
$h(1) = 0.5h(0) + \delta(1) = 0.5$

$h(2) = 0.5h(1) + \delta(2) = (0.5)^2$

$\vdots$

$h(n) = (0.5)^n$ Impulse Response: stable

Example:



$y(n) = \alpha y(n-1) + x(n)$, $n \geq 0$

$y(n) = 0$ for $n < 0$

Difference equation

i) Find the impulse response

ii) Find the step response

i) $x(n) = \delta(n) \Rightarrow y(n) = h(n)$

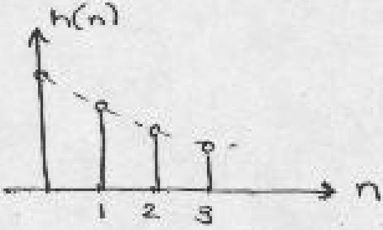
$h(n) = \alpha h(n-1) + \delta(n)$

$$h(n) = \alpha h(n-1) + \delta(n)$$

$n=0$:

$$h(0) = \alpha h(-1) + \delta(0) = 1$$
$$h(1) = \alpha h(0) + \delta(1) = \alpha$$
$$h(2) = - - - - - - = \alpha^2$$
$$\vdots$$
$$h(n) = - - - - - - - - - = \alpha^n$$

$$\boxed{h(n) = \alpha^n u(n)}$$



$\alpha < 1$ (stable)

$$\sum_0^\infty |h(n)| < \infty \Rightarrow \alpha < 1$$

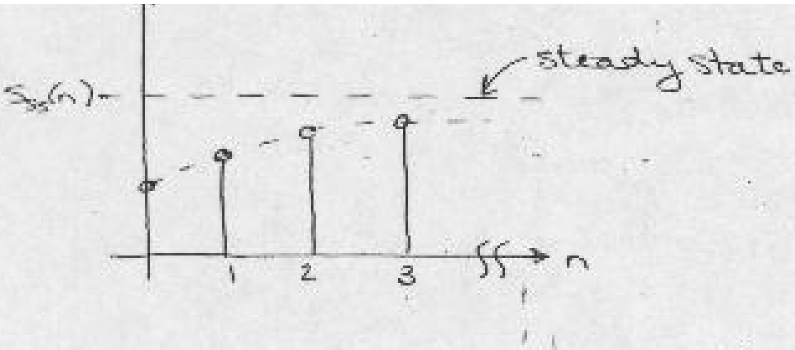
ii) $x(n) = u(n) \Rightarrow y(n) = s(n)$

$$s(n) = \alpha s(n-1) + u(n)$$

$n=0$:

$$s(0) = \alpha s(-1) + u(0) = u(0) = 1$$
$$s(1) = \alpha s(0) + u(1) = 1 + \alpha$$
$$s(2) = 1 + \alpha + \alpha^2$$
$$\vdots$$
$$s(n) = 1 + \alpha + \alpha^2 + \dots + \alpha^n$$
$$= \frac{1-\alpha^{n+1}}{1-\alpha}$$

$$\boxed{s(n) = \frac{1-\alpha^{n+1}}{1-\alpha}}$$



what is n_{ss} when $s(n) = \left(\frac{99}{100}\right) \underbrace{s_n(\infty)}_{\text{steady state value}}$

$$s(n) = \frac{(0.99)(1)}{1-\alpha} = \frac{1-\alpha^{n_{ss}+1}}{1-\alpha}$$

$$0.99 = 1-\alpha^{n_{ss}+1}$$

$$\alpha^{n_{ss}+1} = \frac{1}{100} = 10^{-2}$$

$$\log_{10}\left(\alpha^{n_{ss}+1}\right) = \log_{10} 10^{-2} = -2$$

$$(n_{ss}+1)\log_{10}\alpha = -2$$

$$n_{ss} = \text{ciel}\left(\frac{-2}{\log_{10}\alpha} - 1\right)$$

Note: ciel(2.1) = 3
ciel(2.9) = 3

Example:

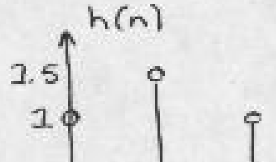
$y(n) = x(n) + 1.5x(n-1) + x(n-2)$

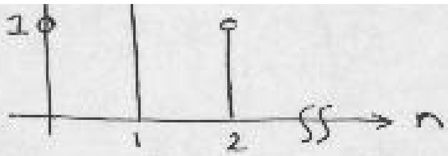
FIR
Non-Recursive Filter

$h(n) = ?$
Network ?

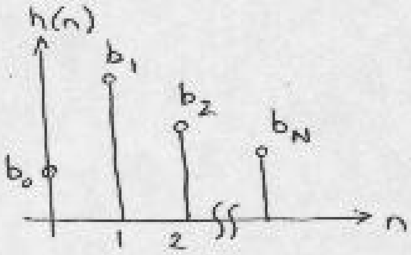
Put $x(n) = \delta(n) \Rightarrow y(n) = h(n)$

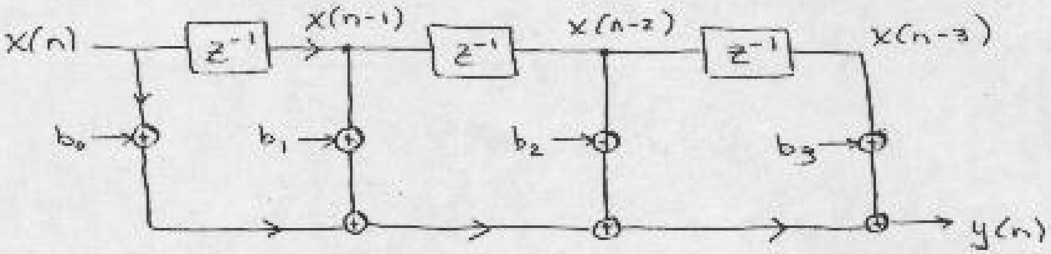
$h(n) = \delta(n) + 1.5\delta(n-1) + \delta(n-2)$





ex: $y(n) = b_0 x(n) + b_1 x(n-1) + \cdots + b_N x(n-M)$





Frequency Response of Discrete-time Systems

$x(n) = e^{j\omega n}$ $\qquad -\infty < n < \infty$

$x(n) = e^{j\omega n} \qquad -\infty < n < \infty$

Note:
$$e^{j\theta} = \cos\theta + j\sin\theta$$
$$\cos\theta = \frac{1}{2}\left\{e^{j\theta} + e^{-j\theta}\right\}$$
$$\sin\theta = \frac{1}{2j}\left\{e^{j\theta} - e^{-j\theta}\right\}$$

<u>Analysis:</u>

$$y(n) = x(n) * h(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k)$$
$$= \sum_{k=-\infty}^{\infty} h(k)\,e^{j\omega(n-k)}$$
$$= \sum_{k=-\infty}^{\infty} h(k)\,e^{j\omega n} e^{-j\omega k} = e^{j\omega n} \sum_{k=-\infty}^{\infty} h(k)\,e^{-j\omega k}$$
$$y(n) = e^{j\omega n} \underbrace{H(e^{j\omega})}_{\text{frequency response}}$$

$$\boxed{H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)\,e^{-j\omega n} < \infty}$$
Frequency Response

Put $x(n) = \cos(\omega n)$:
$$\cos(\omega n) = \frac{1}{2}\left\{e^{j\omega n} + e^{-j\omega n}\right\}$$
$$\cos(\omega n) = \frac{1}{2}e^{j\omega n} + \frac{1}{2}e^{-j\omega n}$$
Response ↓ ⇓ ⇘
$$y(n) = \frac{1}{2}e^{j\omega n} H(e^{j\omega}) + \frac{1}{2}e^{-j\omega n} H(e^{-j\omega})$$

Complex conjugate of $H(e^{j\omega})$:
$$H^*(e^{j\omega}) = \sum h(n)\,e^{j\omega n} = H(e^{-j\omega})$$

$$y(n) = \frac{1}{2}e^{j\omega n} H(e^{j\omega}) + \frac{1}{2}e^{-j\omega n} H(e^{-j\omega})$$

Note: The sum of two complex conjugates equal 2 times

Note: The sum of two complex conjugates equal 2 times the real part of either one (ex: $(a+jb)+(a-jb) = 2a$)

$$\Rightarrow \quad y(n) = \text{real}\left\{ e^{j\omega n} H(e^{j\omega}) \right\}$$

$$H(e^{j\omega}) = \underbrace{\left|H(e^{j\omega})\right|}_{\hookrightarrow \text{magnitude}} e^{j\underbrace{\measuredangle H(e^{j\omega})}_{\hookrightarrow \text{phase}}}$$

$$\begin{aligned} y(n) &= \text{real}\left\{ e^{j\omega n} \cdot \left|H(e^{j\omega})\right| e^{j\measuredangle H(e^{j\omega})} \right\} \\ &= \left|H(e^{j\omega})\right| \text{real}\left\{ e^{j(\omega n + \measuredangle H(e^{j\omega}))} \right\} \\ &= \left|H(e^{j\omega})\right| \cos(\omega n + \measuredangle H(e^{j\omega})) \end{aligned}$$

because $e^{j\theta} = \underbrace{\cos\theta}_{\text{real}} + j\sin\theta$

$$y(n) = \underbrace{\left|H(e^{j\omega})\right|}_{\hookrightarrow \text{mag. resp.}} \cos(\omega n + \underbrace{\measuredangle H(e^{j\omega})}_{\hookrightarrow \text{phase resp.}})$$

If: $x(n) = \alpha \cos(\omega n + \theta)$

Then: $y(n) = \alpha \left|H(e^{j\omega})\right| \cos(\omega n + \theta + \measuredangle H(e^{j\omega}))$

For the frequ. resp. to exist,

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

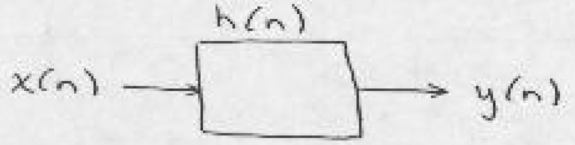
Discrete-time systems are periodic because:

$$e^{j\omega} = e^{j(\omega + 2\pi)} = e^{j(\omega + k2\pi)}$$

$\therefore$ repeat every 2π

Frequency Response of Discrete-time Systems

Frequency Response of Discrete-time Systems



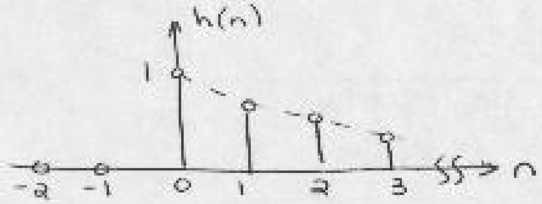
$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} = FT\{h(n)\}$$

If $x(n) = A\cos(\omega_0 n + \phi)$

$$\Rightarrow y_{ss}(n) = A\left|H(e^{j\omega_0})\right|\cos(\omega_0 n + \phi + \measuredangle H(e^{j\omega}))$$

Ex:

Find $H(e^{j\omega})$ for a LTI system with $h(n) = a^n u(n)$
$0 < a < 1$



stable

$\sum_{0}^{\infty} |h(n)| < \infty$ finite

$$H(e^{j\omega}) = \sum_{0}^{\infty} h(n)e^{-j\omega n}$$

$$= \sum_{0}^{\infty} a^n e^{-j\omega n} = \sum_{0}^{\infty} (ae^{-j\omega})^n = \sum_{0}^{\infty} k^n$$

where $k = ae^{-j\omega}$

$$\therefore H(e^{j\omega}) = \frac{1}{1-k} = \frac{1}{1-ae^{-j\omega}}$$

$|k| < 1$

$|ae^{-j\omega}| < 1$

$|a| < 1$

Frequ. Resp:

$$H(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}, \quad a < 1$$

Magnitude Resp.

$$|H(e^{j\omega})| = \frac{1}{|1-ae^{-j\omega}|} = \frac{1}{|1-\{a\cos\omega - ja\sin\omega\}|}$$

$$= \frac{1}{|1-a\cos\omega + ja\sin\omega|}$$

$$|H(e^{j\omega})|^2 = H(e^{j\omega})\,H^*(e^{j\omega})$$

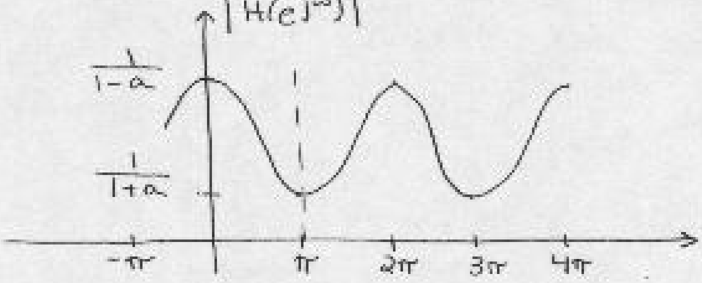
$$= \frac{1}{(1-ae^{-j\omega})} \cdot \frac{1}{(1-ae^{j\omega})}$$

$$= \frac{1}{1+a^2e^0 - a\{e^{j\omega}+e^{-j\omega}\}}$$

$$= \frac{1}{1+a^2-2a\cos\omega}$$

$$\therefore |H(e^{j\omega})| = \frac{1}{\sqrt{1+a^2-2a\cos\omega}}$$

Periodic w/ a period of 2π



Digital Low Pass Filter (look at $0\to\pi$)
(Periodic + Symmetrical)
Even

$$|H(e^{j0})| = \frac{1}{\sqrt{1+a^2-2a}} = \frac{1}{1-a}$$

↳ DC resp. ($\omega=0$)

$$|H(e^{j\pi})| = \frac{1}{\sqrt{1+a^2+2a}} = \frac{1}{1+a}$$

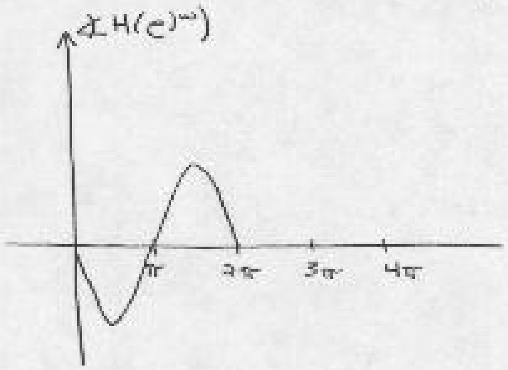
Phase Response

$$\measuredangle H(e^{j\omega}) = \measuredangle \frac{1}{1-ae^{-j\omega}} = \measuredangle 1 - \measuredangle(1-ae^{-j\omega})$$

$$= -\measuredangle(1-ae^{-j\omega}) = -\measuredangle\{1-(a\cos\omega - ja\sin\omega)\}$$

$$= -\measuredangle\{\underbrace{(1-a\cos\omega)}_{\substack{\text{always pos.}\\ \text{because}\\ a<1 \text{ for}\\ \text{stability}}} + \underbrace{ja\sin\omega}_{\substack{\text{pos if } 0<\omega<\pi\\ \text{because sin is}\\ \text{pos. in Quad}\\ \text{I and II}}}\}$$

$$\measuredangle H(e^{j\omega}) = -\tan^{-1}\left(\frac{a\sin\omega}{1-a\cos\omega}\right)$$



(Periodic and Odd)

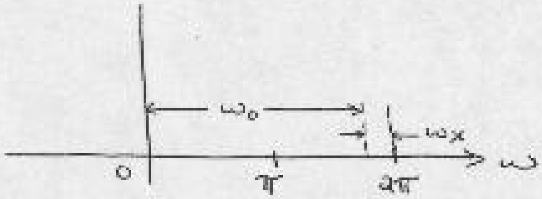
Properties of $H(e^{j\omega})$

1) Continuous function of ω
2) A periodic function of ω, with a period of 2π
   $\Rightarrow H(e^{j\omega}) = H(e^{j(\omega+2k\pi)})$, $k = 0, \pm1, \pm2, \cdots, \infty$
3) magnitude Response is an even function of ω
4) Phase Response is an odd function of ω

Useful frequency range, $0 \leq \omega \leq \pi$

Any $\pi < \omega \leq 2\pi$ in a sequence $x(n) = \cos(\omega n + \phi)$ is going to be reduced to another sequence, $x(n) = \cos(\omega' n + \phi')$, where $0 \leq \omega' \leq \pi$, $\phi' = -\phi$

Assume that $\pi \le \omega_0 \le 2\pi$



$$\omega_0 + \omega_x = 2\pi \Rightarrow \omega_0 = 2\pi - \omega_x$$
$$0 \le \omega_x < \pi$$

$$x(n) = \cos(\omega_0 n + \phi) = \cos((2\pi - \omega_x)n + \phi)$$
$$= \cos(-\omega_x n + \phi + 2\pi n) = \cos(-\omega_x n + \phi)$$
$$= \cos(\omega_x n - \phi)$$

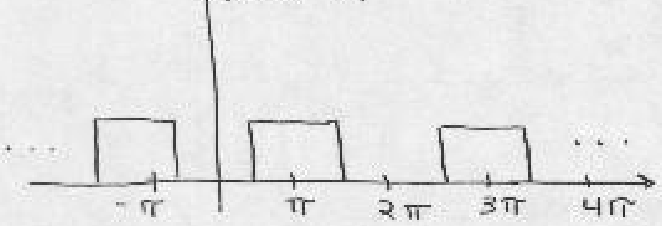
because cos is even

$f_{samp} = 10$ kH

Analog freq		Digital freq.	
f, Hz	Ω	$\omega = \frac{2\pi f}{f_{samp}}$	ϕ
1 kHz		0.2π	same
2 kHz		0.4π	same
4 kHz		0.8π	same
6 kHz		0.8π	$-\phi$
11 kHz		0.2π	same

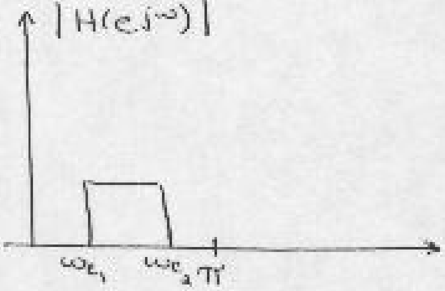
Useful frequ. range: 0 → 5 kHZ
if $f_{samp} = 10$ kHZ

Ideal Digital High Pass Filter:



Periodic

Digital Band Pass Filter:



Periodic

The Fourier Transform of Sequences

x(n)

$$X(e^{j\omega}) = FT\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

Analysis Equation

If: $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$

$$x(n) = FT^{-1}\{X(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

Synthesis Equation

Properties of the Fourier Transform

Consider the following sequences:

$$x(n) \overset{FT}{\longleftrightarrow} X(e^{j\omega})$$

$$y(n) \overset{FT}{\longleftrightarrow} Y(e^{j\omega})$$

1. Linearity

$$ax(n) + by(n) \overset{FT}{\longleftrightarrow} aX(e^{j\omega}) + bY(e^{j\omega})$$

2. Delay or Time-shifting

$$x(n-n_0) \overset{FT}{\longleftrightarrow} X(e^{j\omega})e^{-j\omega n_0}$$

The Fourier Transform of Sequences

$x(n)$

$$X(e^{j\omega}) = FT\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \quad , \quad \sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

$$x(n) = FT^{-1}\{X(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\, d\omega$$

Synthesis Equation

Properties of the Fourier Transform

$$x(n) \xleftrightarrow{FT} X(e^{j\omega})$$

$$y(n) \xleftrightarrow{FT} Y(e^{j\omega})$$

1. Linear Operation

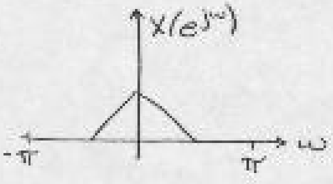
$$\alpha x(n) + \beta y(n) \xleftrightarrow{FT} \alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$$

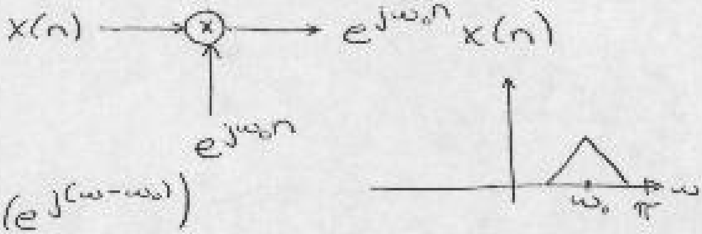
2. Time Shifting

$$x(n-n_0) \xleftrightarrow{FT} e^{-j\omega n_0} X(e^{j\omega})$$

3. Frequency Shifting

$$x(n) \xleftrightarrow{FT} X(e^{j\omega})$$



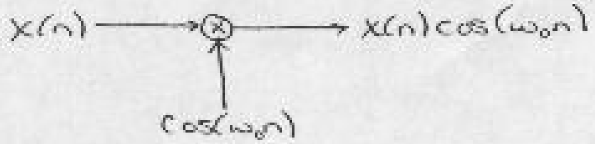


$$x(n)e^{j\omega_0 n} \xleftrightarrow{FT} X(e^{j(\omega-\omega_0)})$$

$$\sum_{n=-\infty}^{\infty} x(n)e^{j\omega_0 n}e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega-\omega_0)n}$$

Special Case: (amplitude modulation)

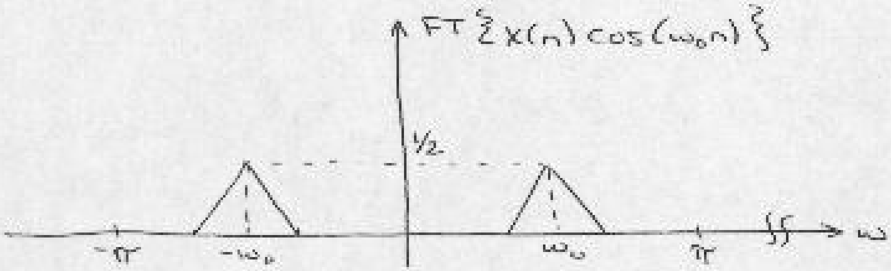
Special Case: (Amplitude Modulation)



$$FT\{x(n)\cos(\omega_0 n)\} = FT\left\{\frac{x(n)}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right)\right\}$$

$$= \frac{1}{2} FT\{x(n)e^{j\omega_0 n}\} + \frac{1}{2} FT\{x(n)e^{-j\omega_0 n}\}$$

$$= \frac{1}{2} X(e^{j(\omega-\omega_0)}) + \frac{1}{2} X(e^{j(\omega+\omega_0)})$$



4. $n x(n) \xleftrightarrow{FT} j\frac{dX(e^{j\omega})}{d\omega}$

$$\sum_{n=-\infty}^{\infty} n x(n) e^{-j\omega n} = ?$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$\frac{d}{d\omega}X(e^{j\omega}) = \frac{d}{d\omega}\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\frac{d}{d\omega}\left(e^{-j\omega n}\right)$$

$$= \sum_{n=-\infty}^{\infty} n x(n) e^{-j\omega n}$$

$$= j\frac{dX(e^{j\omega})}{d\omega}$$

5. Convolution

$$x(n) * y(n) \xleftrightarrow{FT} X(e^{j\omega})Y(e^{j\omega})$$

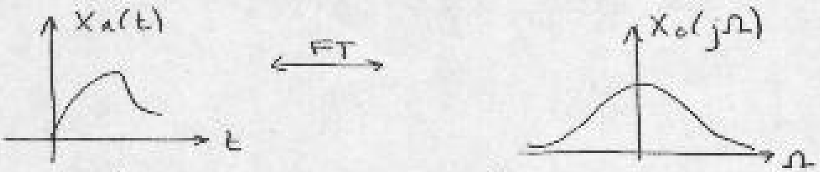
6. Frequency Convolution

$$x(n)y(n) \xleftrightarrow{FT} X(e^{j\omega}) * Y(e^{j\omega})$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})\,Y(e^{j(\omega-\theta)})\,d\theta$$

$= \frac{\;}{2\pi}\int_{-\pi} X(e^{j\theta})\, Y(e^{j(\omega-\theta)})\, d\theta$

7. Parseval's Theorem



$$E=\int_{-\infty}^{\infty}|x_a(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X_a(j\Omega)|^2\,d\omega$$

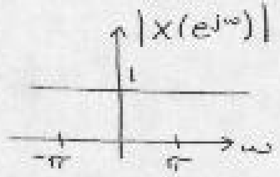
$$X_a(j\Omega)=\int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t}\,dt$$

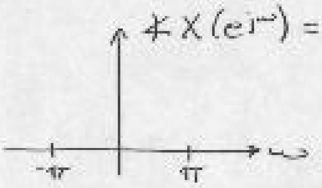
$$E=\underbrace{\sum_{n=-\infty}^{\infty}|x(n)|^2}_{\text{time domain}} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{|X(e^{j\omega})|^2}_{\text{freq. domain}}\,d\omega$$

Examples:

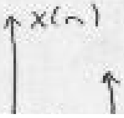
① for $x(n)=\delta(n)$, find $X(e^{j\omega})$

$$X(e^{j\omega})=\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty}\delta(n)e^{-j\omega n} = \delta(0)e^{-j0} = 1$$

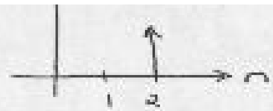




② find $FT\{\delta(n-2)\}$



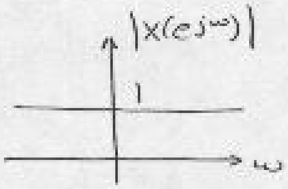
$X(e^{j\omega}) = e^{-j2\omega}\times 1 = e^{-j2\omega}$

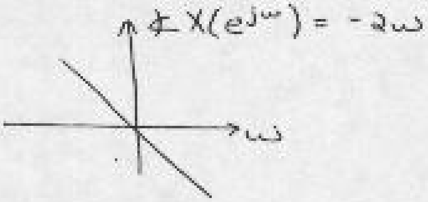


$X(e^{j\omega}) = e^{j} \times 1 = e^{j}$

$|X(e^{j\omega})| = 1$

$\measuredangle X(e^{j\omega}) = -2\omega$

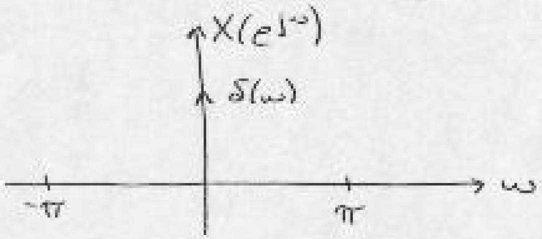




③ $x(n) = a^n u(n), \quad a<1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1-ae^{-j\omega}}$$

④



$x(n) = ?$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\omega)e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\omega)e^{j0}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\omega)\,d\omega = \frac{1}{2\pi} \text{ for all } n$$

In order to get $x(n)=1$, you need to multiply $\delta(\omega)$ by 2π

$$x(n) \xleftrightarrow{FT} 2\pi\delta(\omega)$$

$$x(n)e^{j\omega_0 n} \xleftrightarrow{FT} 2\pi\delta(\omega-\omega_0)$$

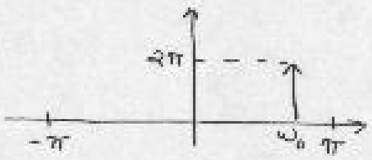
$$x(n) \longleftrightarrow 2\pi\delta(\omega)$$

$$x(n)e^{j\omega_0 n} \xleftrightarrow{FT} 2\pi\delta(\omega-\omega_0)$$

Since $x(n)=1$ for all n

$$e^{j\omega_0 n} \xleftrightarrow{FT} 2\pi\delta(\omega-\omega_0)+\cdots$$

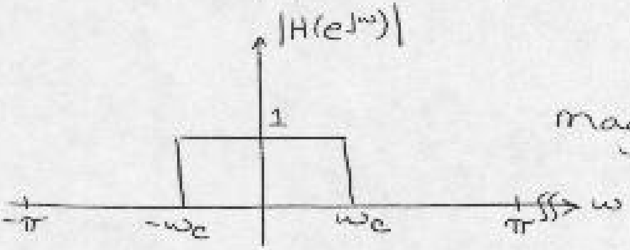
$$= 2\pi\sum_{k=-\infty}^{\infty}\delta(\omega-\omega_0+2k\pi)$$



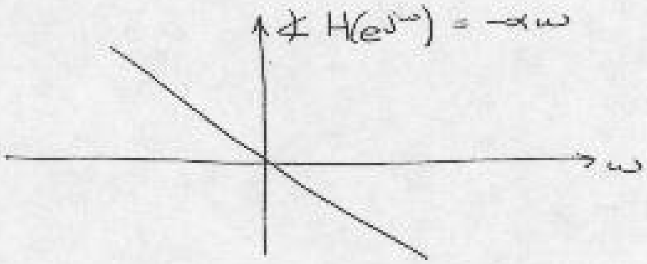
⑤ $\cos\omega_0 n \xleftrightarrow{FT}$?

$$\frac{1}{2}e^{j\omega_0 n}+\frac{1}{2}e^{j\omega_0 n} \xleftrightarrow{FT} \pi\left\{\delta(\omega-\omega_0+2k\pi)+\sum_{k=-\infty}^{\infty}\delta(\omega+\omega_0+2k\pi)+\cdots\right\}$$

⑥ Is an ideal digital low pass filter causal?



magnitude



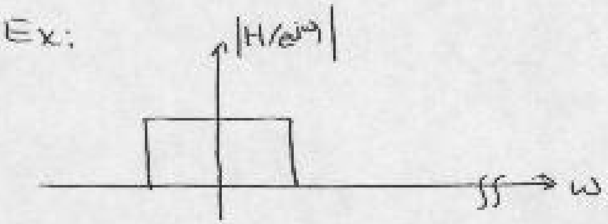
Linear Phase

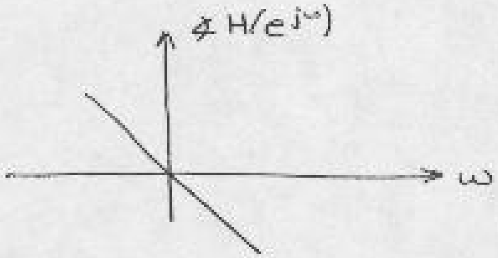
for $-\pi \le \omega \le \pi$

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\measuredangle H(e^{j\omega})} = \begin{cases} 1\times e^{-j\alpha\omega} & |\omega|\le\omega_c \\ 0 & \text{o.w.} \end{cases}$$

Non-Causal

Ex:





for $-\pi \leq \omega \leq \pi$

$$|H(e^{j\omega})| = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{o.w.} \end{cases}$$

$$\measuredangle H(e^{j\omega}) = -\alpha\omega$$

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\measuredangle H(e^{j\omega})} = \begin{cases} e^{-j\alpha\omega} & |\omega| < \omega_c \\ 0 & \text{o.w.} \end{cases}$$

$$h(n) = FT^{-1}\{H(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\alpha\omega} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\alpha)}\, d\omega$$

$\underset{\sim\sim}{n=\alpha}$ (integer)

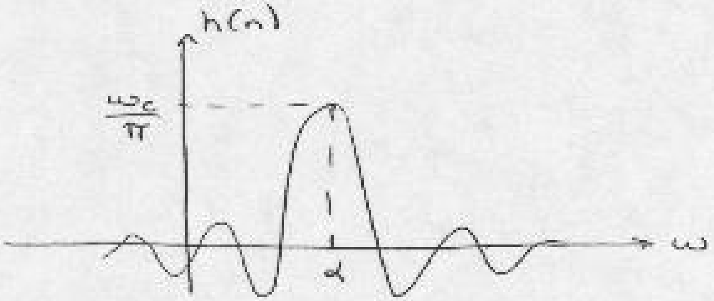
$$h(\alpha) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi}$$

$\underset{\sim\sim}{n\neq\alpha}$

$$h(n) = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi(n-\alpha)j}\left(e^{j\omega_c(n-\alpha)} - e^{-j\omega_c(n-\alpha)}\right) = \frac{\sin\omega_c(n-\alpha)}{\pi(n-\alpha)}$$

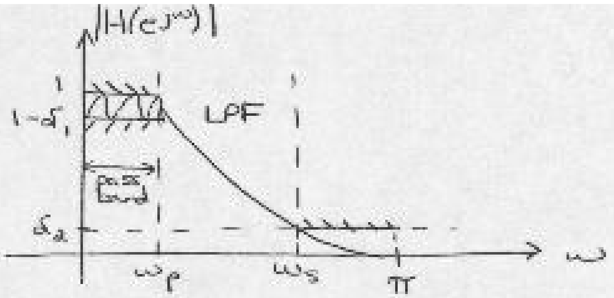
$$h(n) = \begin{cases} \dfrac{\sin\omega_c(n-\alpha)}{\pi(n-\alpha)} & n\neq\alpha \\ \dfrac{\omega_c}{\pi} & n=\alpha \end{cases}$$

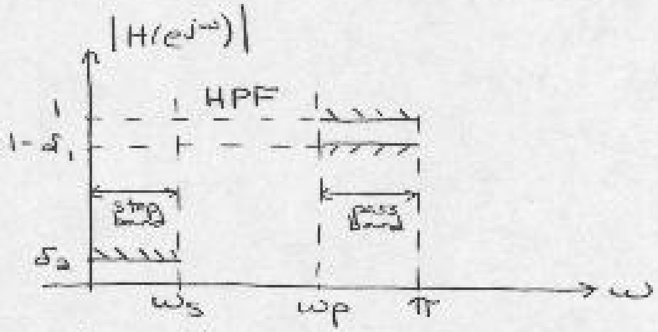


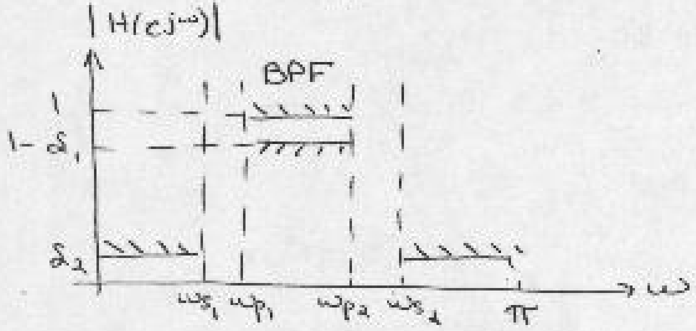
IIR

Non-causal

* All ideal filters are non-causal







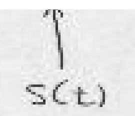
Finding the coefficients F_n, $n=0,\pm 1,\pm 2,\dots$

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} s(t)\, e^{jn\Omega_s t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{jn\Omega_s t}\, dt = \frac{1}{T}\int \delta(t)\, dt = \frac{1}{T}$$

$$\boxed{F_n = \frac{1}{T} \text{ for all } n}$$

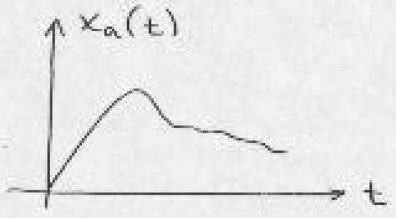
$$x_a(t) \longrightarrow \otimes \longrightarrow x_{as}(t) = \frac{1}{T}\, x_a(t) \sum_{n=-\infty}^{\infty} e^{jn\Omega_s t}$$

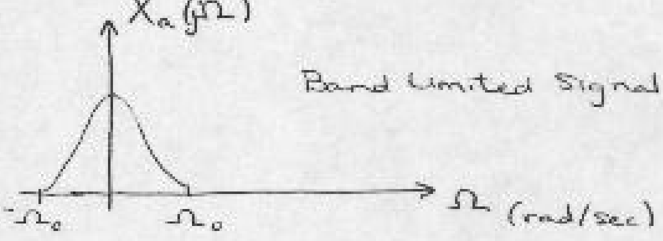


$$S(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\Omega_s t}$$

$$\begin{aligned}
X_{as}(j\Omega) &= FT\left\{x_{as}(t)\right\} \\
&= FT\left\{\frac{1}{T}x_a(t)\sum_{n=-\infty}^{\infty} e^{jn\Omega_s t}\right\} \\
&= \frac{1}{T}FT\left\{\sum_{n=-\infty}^{\infty} x_a(t)e^{jn\Omega_s t}\right\} \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty} FT\left\{x_a(t)e^{jn\Omega_s t}\right\} \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty} X_a(j(\Omega - n\Omega_s)) \\
&= \frac{1}{T}\left\{X_a(j\Omega) + X_a(j(\Omega-\Omega_s)) \right. \\
&\qquad \left. + X_a(j(\Omega-2\Omega_s))\right\}
\end{aligned}$$

Sampling of Continuous Time (Analog) Signals

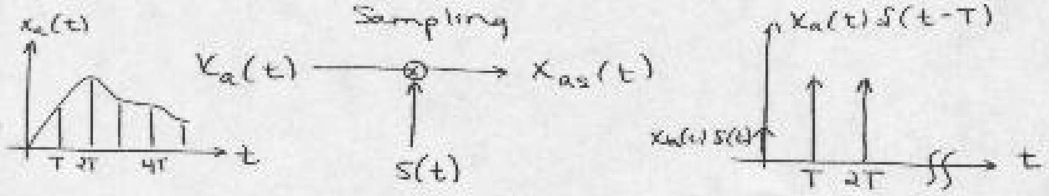


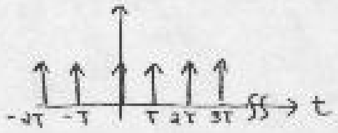


$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t}\,dt$$

if $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$ (finite)

$|X_a(j\Omega)| = \delta$ for $|\Omega| > \Omega_0$





$$s(t) = \delta(t) + \delta(t-T) + \delta(t-2T) + \cdots + \delta(t+T) + \delta(t+2T) \cdots$$
$$= \sum_{k=-\infty}^{\infty} \delta(t-kT) \quad \text{periodic}$$

T is the period

$\Omega_s = \frac{2\pi}{T}$ (fundamental frequ. in rad/sec)

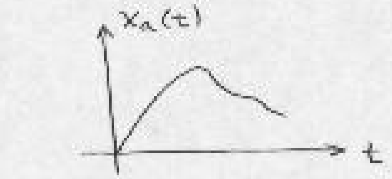
$$\boxed{s(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT) = \sum_{n=-\infty}^{\infty} F_n e^{jn\Omega_s t}}$$

$n=0 \rightarrow$ DC

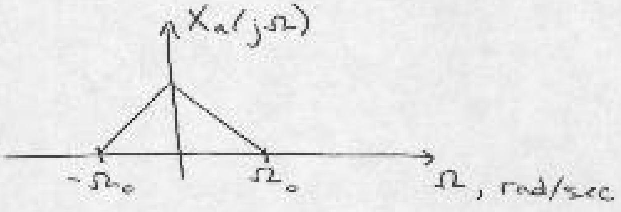
$n=\pm1 \Rightarrow$ fundamental

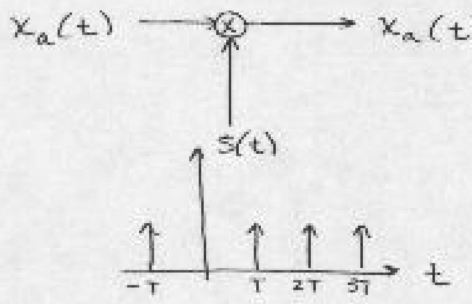
$n=\pm2 \Rightarrow$ 2nd harmonic

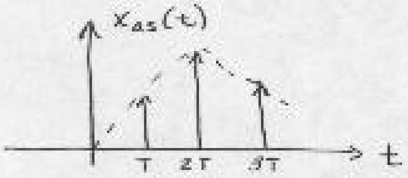
Sampling Theorem



Bandlimited Signal

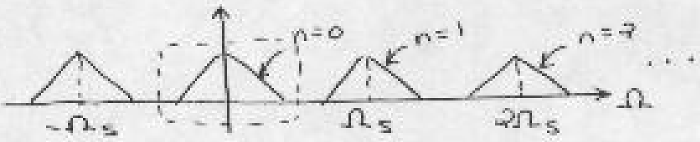


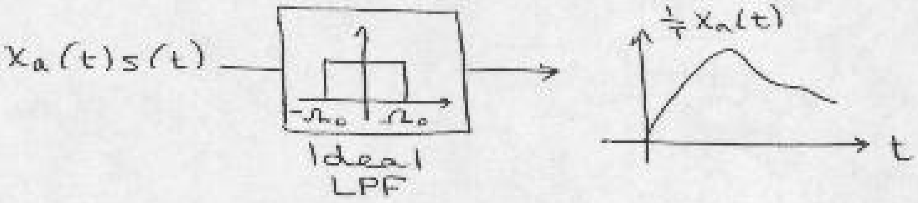




$$X_{as}(j\Omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_a(j(\Omega - n\Omega_s))$$

$$= \frac{1}{T}\left\{X_a(j\Omega) + X_a(j(\Omega - \Omega_s))\right. ;$$





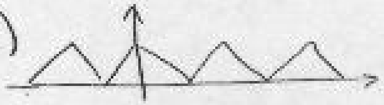
Sampling Frequency Requirements

1) No Aliasing

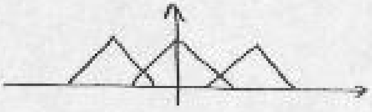
$\Omega_s > 2\Omega_o$

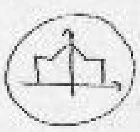
$f_s > 2f_o$

2) $\Omega_s = 2\Omega_o$ (limiting case)



3) $\Omega_s < 2\Omega_o$ (Aliasing + Distortion)

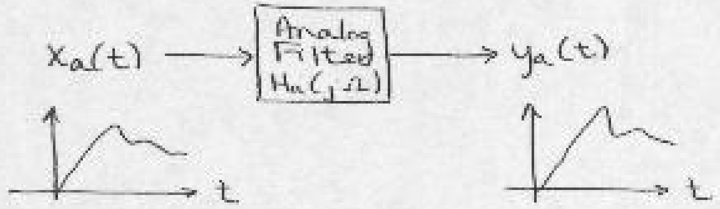




If the signal is not band limited you

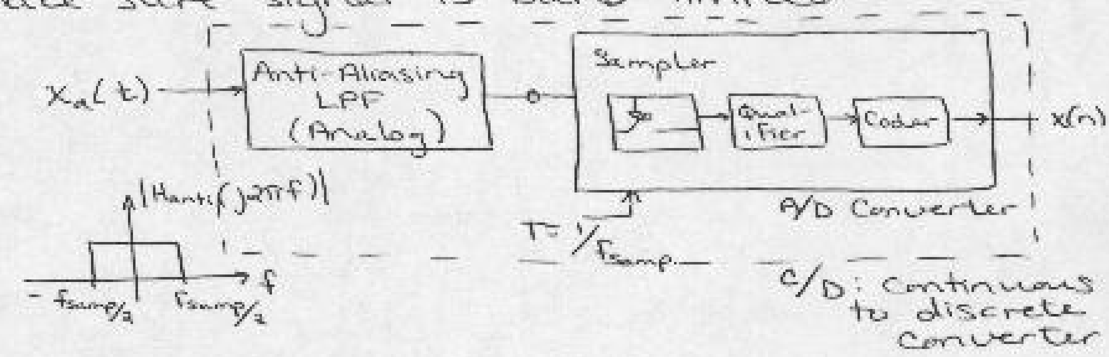
If the signal is not band limited you get overlapping

Processing of Analog (Continuous-time) Signals Using Digital Filters

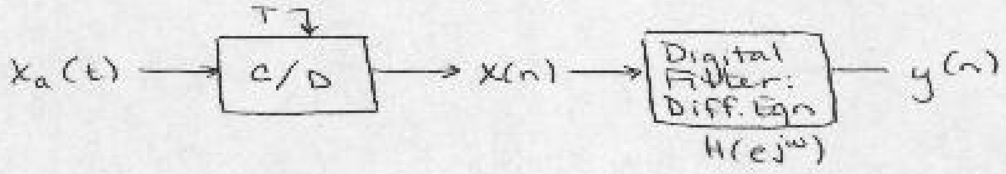


How do you replace analog filter w/ digital filter + components?

- make sure signal is band-limited

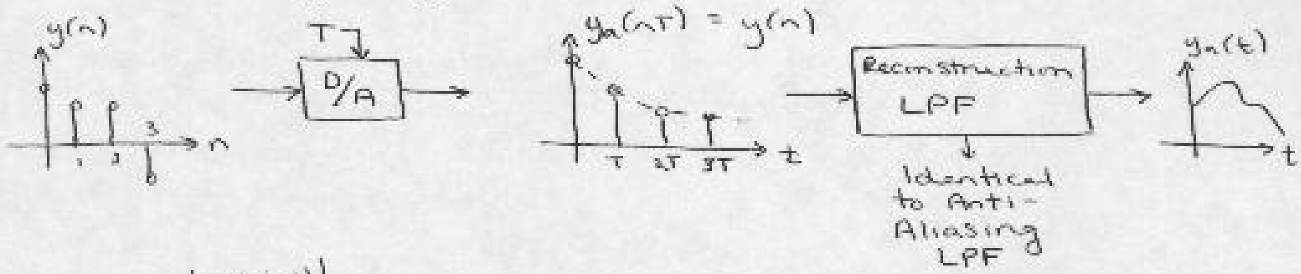


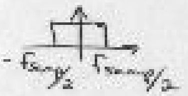
$$x(n) = x_a(nT), \quad n = 0, \pm 1, \dots$$

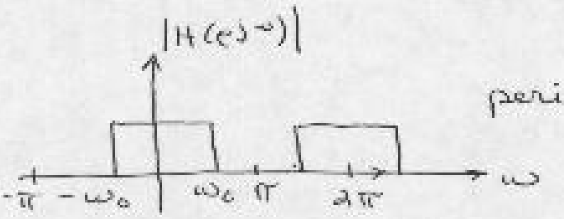


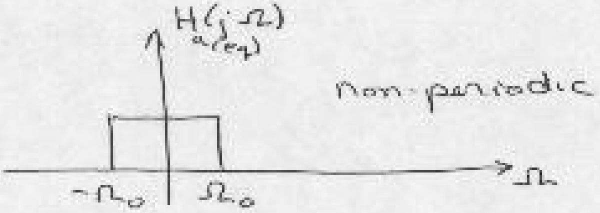
How do you convert back to an analog signal?

analog signal:









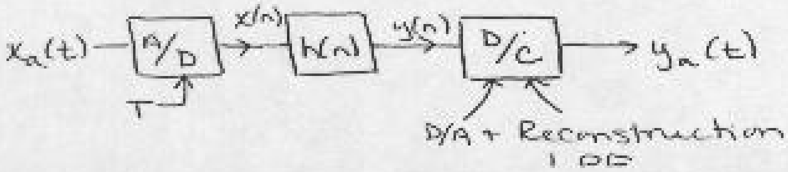
$$\omega_0 = \Omega_0 T = \frac{\Omega_0}{f_{samp}}$$

$$\boxed{\omega = \Omega T}$$

$$H_{a(eq)}(j\Omega) = \begin{cases} H(e^{j\omega}) = |H(e^{j\Omega T})| & , |\Omega| < \pi/f_{samp} \\ 0 & , \text{o.w.} \end{cases}$$

Example:

Consider the system shown below;
The analog signal $x_a(t)$ is connected to the A/D converter without an anti-aliasing LPF;
The digital filter has an impulse response given by $h(n) = (0.5)^n u(n)$;
Assume a sampling period, $T = 0.01$ sec.



Assume $x_a(t) = 3\cos 2\pi t$

a) Find $y_{a(ss)}(t)$ (steady state output)

b) Find two other input signals with different frequencies that will give the same output as Part a)

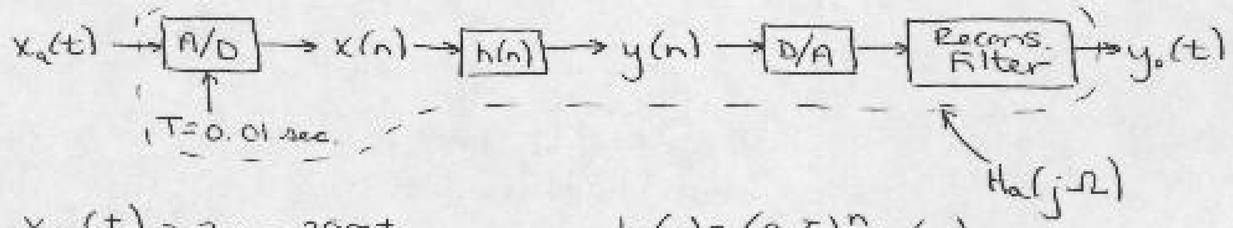
$f_{samp} = 100$ Hz

$f_1 = 10$ Hz

$f_{12} = 110$ Hz

210 Hz

Example: (cont.)



a) $x_a(t) = 3\cos 20\pi t$

$f = 10$ Hz.

$f_{samp} = \frac{1}{T} = 100$ Hz

$h(n) = (0.5)^n u(n)$

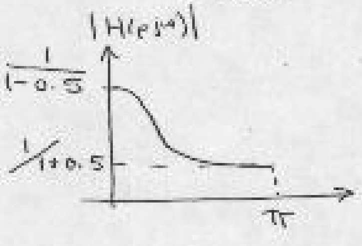
$H(e^{j\omega}) = \frac{1}{1-0.5e^{j\omega}}$

$$\omega_0 = \frac{2\pi f_0}{f_{samp}} = 0.2\pi \text{ rad/sec.}$$

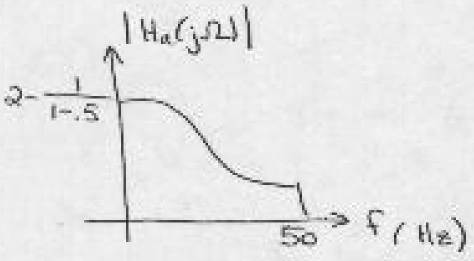
$$y_{ss}(n) = 3|H(e^{j\omega})| \cos(\omega_0 n + \measuredangle H(e^{j\omega}))$$
$$= 4.518 \cos(\omega_0 n - 0.458)$$
$$\therefore\ y_a(t)\Big|_{ss} = 4.518 \cos(20\pi t - 0.458)$$

b) To avoid aliasing, put $x_a(t)$ through a LPF w/ a cutoff frequency, $f_c = 50$ Hz, first, before putting it through the A/D

$$H(e^{j\omega}) = \frac{1}{1-0.5e^{-j\omega}} \qquad \frac{1}{|1-0.5e^{-j\omega}|} = |H(e^{j\omega})|$$



periodic



$\omega = \frac{2\pi f}{f_{samp}}$ $\qquad \Omega = 2\pi f$

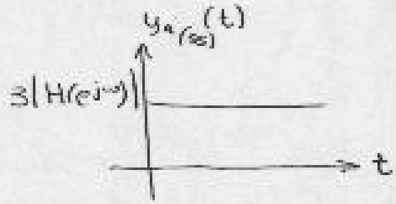
$$H_a(j\Omega) = \begin{cases} H(e^{j2\pi f/f_{samp}}) & , \quad 0 \le f \le 50 \text{ Hz.} \\ 0 & , \quad f > 50 \text{ Hz.} \end{cases}$$

c) Find the steady state for $x_a(t) = 3u_a(t)$

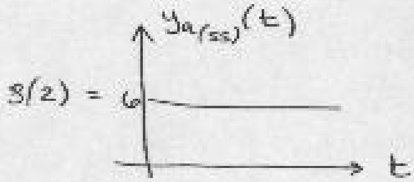
d) Find the diff. eqn. of the digital filter

i.e., $y_{a_{(ss)}}(t) = ??$

c)



dc or $\Omega = 0 \Rightarrow \omega = 0$



$y_{a_{(ss)}}(t) = 6$

d) $$h(n) \xrightarrow{FT} H(e^{j\omega}) = \frac{1}{1-0.5e^{-j\omega}} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$Y(e^{j\omega}) - 0.5e^{-j\omega}Y(e^{j\omega}) = X(e^{j\omega})$$

$$\Big\downarrow FT^{-1} \qquad \Big\downarrow FT^{-1} \qquad \Big\downarrow FT^{-1}$$

$$y(n) - 0.5\,y(n-1) = x(n)$$

<u>General Case</u>:

Given $$H(e^{j\omega}) = \frac{b_0 + b_1e^{-j\omega} + b_2e^{-j2\omega}}{\underset{a_0}{1} + a_1e^{-j\omega} + a_2e^{-j2\omega}} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

Diff. Equ.

$$y(n) + a_1y(n-1) + a_2y(n-2) = b_0x(n) + b_1x(n-1) + b_2x(n-2)$$

Matlab:

Ex: $f_1(x) = 3x^2 + 4x - 1$

$f_2(x) = 10x^2 + 2x + 5$

$g(x) = f_1(x) \cdot f_2(x) = 30x^4 + 46x^3 - \cdots - 5$

```
> f1 = [3  4  -1];
  f2 = [10  2  5];
  g = conv(f1, f2);        % g = [30  46  13  18  -5]
```

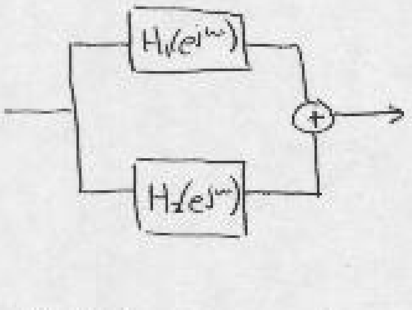
Ex: $(f(x))^8$

```
> f12 = conv(f1, f1);
  f14 = conv(f12, f12);
  g = conv(f14, f14);
```

Ex: $$H(e^{j\omega}) = \frac{\overbrace{(1+2e^{-j\omega}+e^{-j2\omega})}^{b_1}\overbrace{(2-4e^{-j\omega}+10e^{-j2\omega})}^{b_2}}{(\cdots\cdots)(\cdots\cdots)}$$

```
> b1 = [1  2  1];
  b2 = [2  -4  10];
  b = conv(b1, b2);
```

$H(e^{j\omega}) = H_1(e^{j\omega}) H_2(e^{j\omega})$



$H(e^{j\omega}) = H_1(e^{j\omega}) + H_2(e^{j\omega})$

$H(e^{j\omega}) = \frac{1}{\quad} + \frac{1}{\quad}$

$$H(e^{j\omega}) = \frac{1}{1-0.5e^{-j\omega}} + \frac{1}{1-0.4e^{-j\omega}}$$

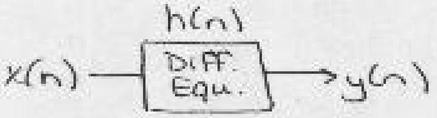
$$= (1-0.4e^{-j\omega}) + (1-0.5e^{-j\omega})$$

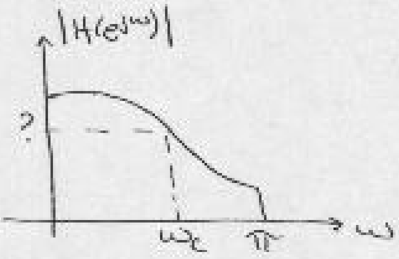
$$H(e^{j\omega}) = \frac{2-0.9e^{-j\omega}}{1-0.9e^{-j\omega}+0.2e^{-j2\omega}}$$

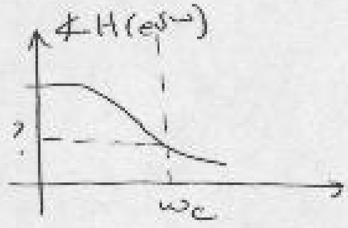
$\downarrow$

Diff. Equ.:

$$y(n) - 0.9y(n-1) + 0.2y(n-2) = 2x(n) - 0.9x(n-1)$$



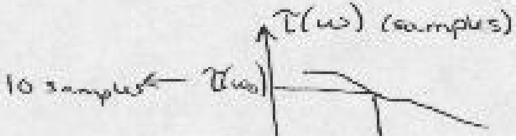


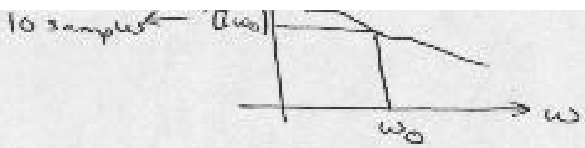


$$x(n) = \cos(\omega_0 n + \pi/4) \longrightarrow y_{ss}(n) = 1 \times |H(e^{j\omega})| \cos(\omega_0 n + \tfrac{\pi}{4} + \measuredangle H(e^{j\omega})$$

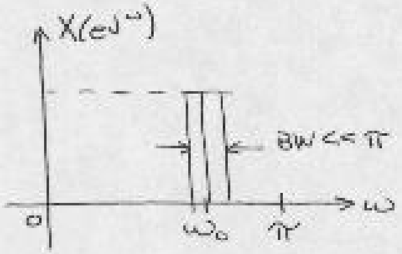
$$\tau(\omega) = -\frac{d\measuredangle H(e^{j\omega})}{d\omega}$$

$\hookrightarrow$ group delay





Synthesize a narrow-band signal centered around ω_0



To create a narrow band signal, add a series of sine waves:

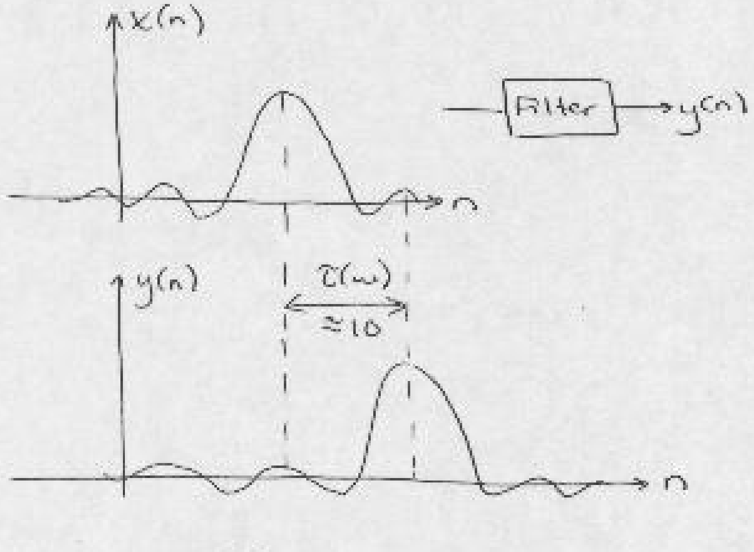
ω_0

$\omega_0 + \pi/128$

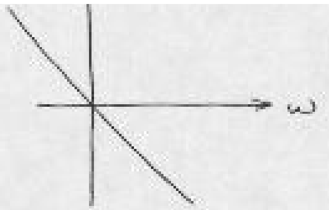
$\omega_0 + (\pi/128)(2)$

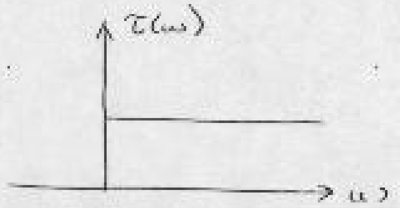
$\omega_0 - \pi/128$

etc.



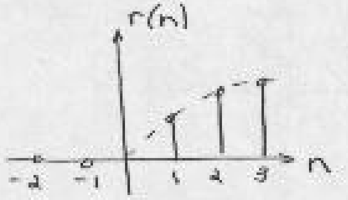
$\angle H(e^{j\omega}) = -5\omega$





2-3

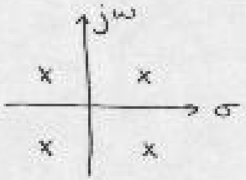
The Z-Transform



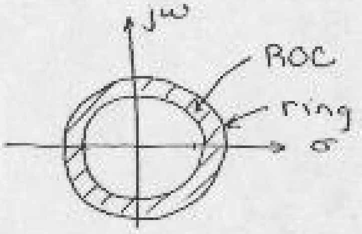
$R(e^{j\omega}) = FT\{r(n)\}$
does not exist

✱ $x(n) \xleftrightarrow{z} X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$ given that $\sum_{n=-\infty}^{\infty} |x(n)||z^{-n}| < \infty$

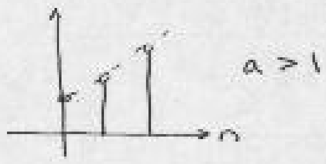
z is a complex number $z = \sigma + j\omega$

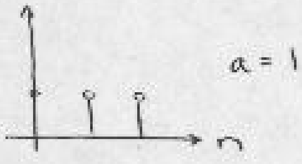


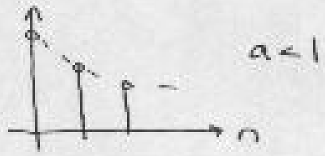
z-plane



Example:
Find the Z-transform of $x(n) = a^n u(n)$







$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \underbrace{(az^{-1})}_{K}{}^n$$

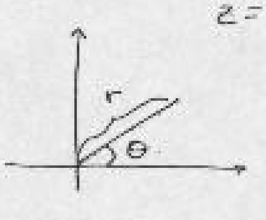
$$= \frac{1}{1-az^{-1}}, \quad \underbrace{|az^{-1}|<1}_{ROC}$$

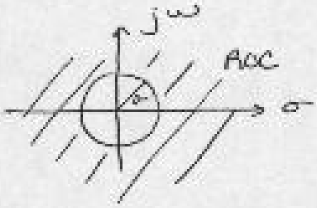
Note: $\sum_{n=0}^{\infty} k^n = \frac{1}{1-k}, \quad |k|<1$

$$\frac{a}{|z|}<1$$

$$\Rightarrow |z|>a \Rightarrow r>a$$

$z = re^{j\theta}$





$x(n) = a^n u(n)$

☆ Special Cases:

(i) $a = 1 \Rightarrow u(n)$

$$\therefore \mathcal{Z}(u(n)) = \frac{1}{1-z^{-1}}$$

(ii) $x(n) = e^{j\omega n} u(n)$

$a = e^{j\omega}$

$$X(z) = \frac{1}{1-e^{j\omega}z^{-1}}$$

(iii) $x(n) = (\cos(\omega n))\, u(n)$

$$= \frac{1}{2}\{e^{j\omega n} + e^{-j\omega n}\} u(n)$$

$$= \frac{1}{2} e^{j\omega n} u(n) + \frac{1}{2} e^{-j\omega n} u(n)$$

$$\therefore X(z) = \frac{1}{2} \cdot \frac{1}{1-e^{j\omega}z^{-1}} + \frac{1}{2} \cdot \frac{1}{1-e^{-j\omega}z^{-1}}$$

$$= \frac{1}{2}\left(\frac{1-e^{-j\omega}z^{-1}+1-e^{j\omega}z^{-1}}{(1-e^{j\omega}z^{-1})(1-e^{-j\omega}z^{-1})}\right)$$

$$\frac{}{2}\left(\frac{}{(1-e^{j\omega}z^{-1})(1-e^{-j\omega}z^{-1})}\right)$$

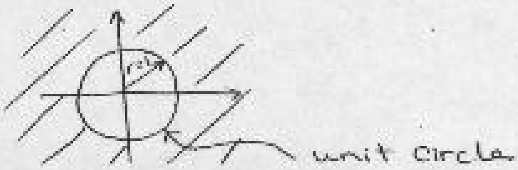
$$= \frac{1}{2}\left(\frac{2-z^{-1}\{e^{j\omega}+e^{-j\omega}\}}{1-z^{-1}\{e^{j\omega}+e^{-j\omega}\}+z^{-2}}\right)$$

$$= \frac{1-\cos\omega z^{-1}}{1-2\cos\omega z^{-1}+z^{-2}}$$

$|z| = r > |a|$

$= r > |e^{j\omega}|$

$|r| > 1 \Rightarrow$ ROC



(iv) $x(n) = \delta(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} = 1$$

ROC: entire z-domain plane

☆ Properties of the z-transform

1) Linearity

$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$, ROC_1

$y(n) \overset{\mathcal{Z}}{\longleftrightarrow} Y(z)$, ROC_2

$$\alpha x(n) + \beta y(n) \overset{\mathcal{Z}}{\longleftrightarrow} \alpha X(z) + \beta Y(z)$$

ROC: $ROC_1 \cap ROC_2$

↖ intersection

2) Delay

$$x(n-n_0) \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0} X(z)$$

↳ integer

3) Complex Scale Change

$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$

$$\omega^{-n} x(n) \overset{\mathcal{Z}}{\longleftrightarrow} \sum_{n=-\infty}^{\infty} \omega^{-n} x(n) z^{-n}$$

$$\longrightarrow \sum_{n=-\infty} \omega \quad x(n)\, z$$

$$= \sum_{n=-\infty}^{\infty} x(n)(\omega z)^{-n} = X(\omega z)$$

ex: $x(n) = \cos\omega_0 n\, u(n)$

$$X(z) = \frac{1 - \cos\omega_0 z^{-1}}{1 - 2\cos\omega_0 z^{-1} + z^{-2}}$$

$$\mathcal{Z}\{a^n \cos\omega_0 n\, u(n)\} = \mathcal{Z}\{\omega^{-n} \cos\omega_0 n\, u(n)\}$$

$$\Rightarrow X(z/a) \qquad\qquad \omega = a^{-1}$$

4) Complex Differentiation

$$x(n) \xleftrightarrow{\mathcal{Z}} X(z)$$

$$\mathcal{Z}\{n\, x(n)\} = ?$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\frac{dX(z)}{dz} = \frac{d}{dz} \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n) \frac{d}{dz} z^{-n}$$

$$= -\sum_{n=-\infty}^{\infty} n\, x(n) z^{-n-1} = -z^{-1} \sum_{n=-\infty}^{\infty} n\, x(n) z^{-n}$$

$$= -z^{-1} \mathcal{Z}\{n\, x(n)\}$$

$$\therefore \quad \mathcal{Z}\{n\, x(n)\} = -z \frac{dX(z)}{dz}$$

Example:

Find the z-transform of the unit ramp sequence

$r(n) = n\, u(n)$

$$\mathcal{Z}\{u(n)\} = \frac{1}{1 - z^{-1}} = \frac{z}{z-1} = U(z)$$

$$\therefore \mathcal{Z}\{n\, u(n)\} = \mathcal{Z}\{r(n)\} = -z \frac{d}{dz}(U(z))$$

$$= -z \frac{d}{dz} \left\{ \frac{z}{z-1} \right\}$$

$$= -z\frac{d}{dz}\left\{\frac{z}{z-1}\right\}$$

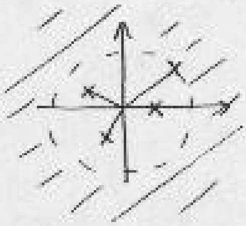
$$= -z\left\{\frac{(z-1)-z}{(z-1)^2}\right\} = \frac{-z(-1)}{(z-1)^2} = \frac{z}{(z-1)^2}$$

ROC: $|z|>1$

Poles of $R(z)$: 1

* The ROC can <u>not</u> include any of the poles of $X(z)$

Poles of $X(z)$: $z_{p_1}, z_{p_2}, \cdots, z_{p_n}$



* ROC: $|z| > \max\{|z_{p_1}|, |z_{p_2}|, \cdots, |z_{p_n}|\}$

Assume that $x(n)=0$ for $n<0$

$\Rightarrow$ causal or right-handed sequences

<u>ex:</u> $X(z) = \frac{z}{(z-\frac{1}{2})(z-\frac{1}{4})} \Rightarrow z_{p_1} = \frac{1}{2}, \quad z_{p_2} = \frac{1}{4}$

$\Rightarrow$ ROC: $|z| > \frac{1}{2}$

5) Convolution

$x(n) \xleftrightarrow{\mathcal{Z}} X(z)$

$y(n) \xleftrightarrow{\mathcal{Z}} Y(z)$

$$\mathcal{Z}(x(n) * y(n)) = X(z)Y(z)$$

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n)$$

$$\downarrow\mathcal{Z} \qquad \downarrow\mathcal{Z}$$

$$Y(z) = X(z)H(z)$$

$$H(z) = \mathcal{Z}\{h(n)\} = \frac{Y(z)}{X(z)}$$

$\hookrightarrow$ Transfer

$X(z)$

↳ Transfer Function

Diff. Eqn:

$$y(n) + a_1 y(n-1) + a_2 y(n-2)$$
$$= b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$$

Find $H(z)$:

Take z-transform of both sides

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z)$$
$$= b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z)$$

$$Y(z)\{1 + a_1 z^{-1} + a_2 z^{-2}\} = X(z)\{b_0 + b_1 z^{-1} + b_2 z^{-2}\}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\{b_0 + b_1 z^{-1} + b_2 z^{-2}\}}{\{1 + a_1 z^{-1} + a_2 z^{-2}\}}$$

2-8

Properties of the Z-Transform (cont.)

6. Initial Value Theorem

If $x(n) = 0$ for $n < 0$ (causal or RH sequence)

$x(0) = ?$

↳ initial value

$$X(z) = \sum_{-\infty}^{\infty} x(n) z^{-n} = \sum_{0}^{\infty} x(n) z^{-n} = x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} \ldots$$

$$x(0) = \lim_{z \to \infty} X(z)$$

ex: $U(z) = \dfrac{1}{1 - z^{-1}}$

$$\lim_{z \to \infty} U(z) = 1$$

$$\lim_{z\to\infty} U(z) = 1$$

7. Final Value Theorem

$$\lim_{n\to\infty} x(n) = \lim_{z\to 1}\left\{\frac{z-1}{z} X(z)\right\}$$

It is assumed that the poles of $X(z)$ are all inside the unit circle

Applications of the z-Transform

$$y(n) = x(n) * h(n)$$
$$Y(z) = X(z)\,H(z)$$

$$H(z) = \mathcal{Z}\{h(n)\} = \frac{Y(z)}{X(z)}$$

↳ Transfer Function

$$a_0 y(n) + a_1 y(n-1) + a_2 y(n-2) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$$

Take $\mathcal{Z}$-Transform of both sides:

$$a_0 Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z)$$

$$Y(z)\{a_0 + a_1 z^{-1} + a_2 z^{-2}\} = X(z)\{b_0 + b_1 z^{-1} + b_2 z^{-2}\}$$

★ $$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{a_0 + a_1 z^{-1} + a_2 z^{-2}}$$

a_0 ↳ 1

$$H(z) = \frac{K(z-z_1)(z-z_2)\cdots}{(z-z_{p_1})(z-z_{p_2})\cdots}$$

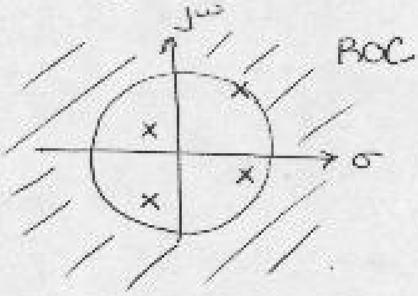
Poles: $z_{p_1}, z_{p_2}, \cdots, z_{p_n}$

Zeros: $z_1, z_2, \cdots, z_n$

$$H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$h(n) = 0 \quad \text{for} \quad n < 0$$

Note: The ROC of $H(z)$ does not include any Pole of $H(z)$



ROC : $r > \max\{|z_{p_1}|, |z_{p_2}|, \cdots\}$

Outside a circle which includes all of the Poles

★ Stability & Region of Convergence

Theorem: A LTI system with transfer function $H(z)$ is stable if and only if the region of convergence of $H(z)$ contains the unit circle (Causal system)

For a system to be stable:

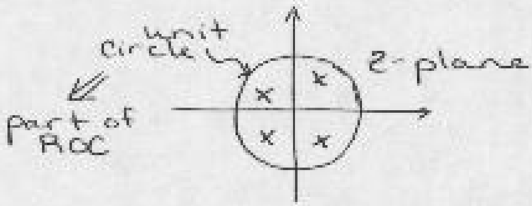
$$\sum_{0}^{\infty} |h(n)| < \infty \quad \text{(finite)} \Rightarrow |z| = 1 \text{ is in the ROC}$$

$$H(z) = \mathcal{Z}\{h(n)\} = \sum_{0}^{\infty} h(n) z^{-n}$$

$$\text{if} \sum_{n=0}^{\infty} |\underbrace{h(n) z^{-n}}_{\to \text{ROC}}| < \infty$$

$$\geq \sum_{n=0}^{\infty} |h(n)||z^{-n}| < \infty$$

$n=0$



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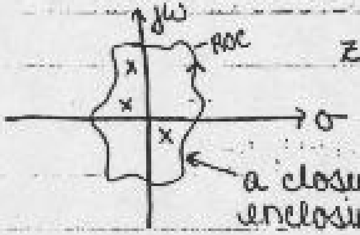
Finding the Inverse z-transform

☆ $x(n) = \mathcal{Z}^{-1}\{X(z)\}$

$$= \frac{1}{2\pi j}\oint X(z) z^{n-1} dz$$



z-plane

a closed path c enclosing origin & poles of X(z)

other methods

☆ - using power series expansion

given X(z) which is causal sequence

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots$$

Example:

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

long division:

$$1+1.5z^{-1}+1.75z^{-2}+\ldots$$

$$1-1.5z^{-1}+0.5z^{-2} \,\overline{)\;1\;}$$

denominator numerator

$$-\;\underline{1-1.5z^{-1}+0.5z^{-2}}$$

$$1.5z^{-1}-0.5z^{-2}$$

$$-\;\underline{1.5z^{-1}-2.25z^{-2}} + \ldots$$

$$+1.75z^{-2}$$

$$\therefore X(z) = 1+1.5z^{-1}+1.75z^{-2}+\ldots$$

$x(0)=1$

$x(1)=1.5$

$x(2)=1.75$

⋮

Example:

Find inverse z-transform of $X(z) = \log(1+az^{-1})$, $|z|>|a|$

note: $\log(1+x) = \sum^{\infty}(-1)^{n+1} x^{n} \ldots x - x^{2} \; x^{3}$

note: $\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

$|x| < 1$

Taylor Series Expansion

Let $x = az^{-1}$ $\quad |x|<1 \Longrightarrow |z| > |a|$ ✓

$$\therefore X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

$x(n) = ?$

note: $X(z) = \sum_{0}^{\infty} x(n) z^{-n}$

$x(0) = 0$

$x(n) = (-1)^{n+1} \frac{a^n}{n} \quad n \neq 0$

$$x(n) = \begin{cases} (-1)^{n+1}\left(\frac{a^n}{n}\right) & n \geq 1 \\ 0 & n \leq 0 \end{cases}$$

- using partial fraction expansion

A) Distinct poles case

$$\frac{X(z)}{z} = \frac{num'(z)}{(z-P_1)(z-P_2)\cdots(z-P_n)}$$ different poles

$$= \frac{A_1}{(z-P_1)} + \frac{A_2}{(z-P_2)} + \cdots + \frac{A_n}{(z-P_n)}$$

find A_1:

$$\left.\frac{X(z)}{z}(z-P_1)\right|_{z=P_1} = A_1 + \frac{A_2 \overset{\nearrow 0}{(z-P_1)}}{(z-P_2)} + \cdots + \frac{A_n \overset{\nearrow 0}{(z-P_1)}}{(z-P_n)}$$

$$A_i = \left.\frac{X(z)}{z}(z-P_i)\right|_{z=P_i}$$

$X(z) = A_1 + A_2 + \cdots$

$$\frac{X(z)}{z} = \frac{A_1}{z-P_1} + \frac{A_2}{z-P_2} + \cdots$$

$$X(z) = \frac{A_1 z}{z-P_1} + \frac{A_2 z}{z-P_2} + \cdots$$

$$= \frac{A_1}{1-z^{-1}P_1} + \frac{A_2}{1-z^{-1}P_2} + \cdots$$

$$x(n) = \mathcal{Z}^{-1}\left\{\frac{A_1}{1-P_1 z^{-1}}\right\} + \cdots$$

$$= \{A_1(P_1)^n + A_2(P_2)^n + \ldots\} u(n)$$

Example:

Determine $x(n) = \mathcal{Z}^{-1}\left\{\dfrac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}\right\}$

$$X(z) = \frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$$

$$= \frac{z^2+z}{z^2-z+0.5} \longleftarrow \text{multiply by } z^2$$

$$\frac{X(z)}{z} = \frac{z+1}{z^2-z+0.5}$$

finding poles of $z^2-z+0.5$:

$$P_{1,2} = \frac{1 \pm \sqrt{1-4(0.5)}}{2} = \frac{1 \pm j}{2}$$

$$P_2 = P_1^*$$

$$P_1 = \frac{1+j}{2} = |P_1| e^{j\angle P_1} = \frac{\sqrt{2}}{2} e^{j\pi/4} = \frac{1}{\sqrt{2}} e^{\pi/4 j}$$

$$P_2 = P_1^* = \frac{1}{\sqrt{2}} e^{-j\pi/4}$$

now want: $$\frac{X(z)}{z} = \frac{A_1}{z-P_1} + \frac{A_2}{z-P_2}$$

$$= \frac{A_1}{z-P_1} + \frac{A_2}{z-P_1^*}$$

$$A_1 = \frac{X(z)}{z}(z-P_1)\Big|_{z=P_1}$$

$$A_2 = \frac{X(z)}{z}(z-P_1^*)\Big|_{z=P_1^*} = A_1^*$$

In example: $$\frac{X(z)}{z} = \frac{z+1}{\left(z-\frac{1+j}{2}\right)\left(z-\frac{1-j}{2}\right)}$$

$$A_1 = \frac{1-j3}{2}, \quad A_2 = \frac{1+j3}{2} \qquad A_1: \frac{\left(\frac{1+j}{2}+1\right)}{\left(\frac{1+j}{2}-\frac{1-j}{2}\right)}$$

$$= \frac{\sqrt{10}}{2}e^{-j1.249} \qquad = \frac{\sqrt{10}}{2}e^{j1.249} \qquad = \frac{\frac{1+j+2}{2}}{2j}$$

$$\therefore X(z) = \frac{A_1 z}{z-P_1} + \frac{A_1^* z}{z-P_1^*}$$

$$= \frac{A_1}{1-z^{-1}P_1} + \frac{A_1^*}{1-P_1^*z^{-1}}$$

Note: $x(n) = \{A_1(P_1)^n + A_1^*(P_1^*)^n\}u(n)$

sum of complex conjugate

$(\alpha+j\beta)+(\alpha-j\beta) = 2\alpha \Rightarrow 2(\text{Real part})$

$$x(n) = 2\,\text{Real}\{A_1(P_1)^n\}u(n)$$

$$= 2\,\text{Real}\left\{\frac{\sqrt{10}}{2}e^{j1.249}\left(\frac{1}{\sqrt{2}}e^{j\pi/4}\right)^n\right\}u(n)$$

$$= \sqrt{10}\,\text{Real}\left\{\left(\frac{1}{\sqrt{2}}\right)^n e^{jn\pi/4}\cdot e^{j1.249}\right\}u(n)$$

$$= \sqrt{10}\left(\frac{1}{\sqrt{2}}\right)^n \text{Real}\left\{e^{j(n\pi/4-1.249)}\right\}u(n)$$

$$\therefore x(n) = \sqrt{10}\left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{n\pi}{4} - 1.249\right) u(n)$$

★ B) Multiple Order Poles

ex: $X(z) = \dfrac{1}{(1+z^{-1})(1-z^{-1})^2}$ multiply by z^3

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2}$$

poles: $z_{p1} = -1$

$z_{p2} = z_{p3} = 1$

$$= \frac{A_1}{(z+1)} + \frac{A_2}{(z-1)} + \frac{A_3}{(z-1)^2}$$

$$A_1 = \frac{X(z)}{z}(z+1)\Big|_{z=-1}$$

$$= \frac{z^2}{(z-1)^2}\Big|_{z=-1}$$

$$= \frac{1}{4}$$

$$\frac{X(z)}{z}(z-1)^2 = A_3 + \frac{A_1}{(z+1)}(z-1)^2 + A_2(z-1) \qquad ①$$

$$A_3 = \frac{X(z)}{z}(z-1)^2\Big|_{z=1}$$

$$= \frac{z^2}{z+1}\Big|_{z=1}$$

$$= \frac{1}{2}$$

$\dfrac{d}{dz}$ of both sides of ① $\longrightarrow$

$$\frac{d}{dz}\left\{\frac{X(z)}{z}(z-1)^2\right\}\Big|_{z=1}$$

$$\frac{d}{dz}\left\{\frac{z^2}{z+1}\right\}\Bigg|_{z=1} = \cancel{\frac{dA_3}{dz}}^{\,0} + A_2 + \cancel{(\ldots)(z-1)}^{\,0}$$

$$A_2 = \frac{d}{dz}\left\{\frac{z^2}{z+1}\right\}\Bigg|_{z=1} = \frac{2z(z+1) - z^2}{(z+1)^2}$$

$$= \frac{4}{8}\ \frac{3}{4} \qquad = \frac{2(2)-1}{2^2} = \frac{3}{4}$$

Thus, $X(z) = \frac{1/4}{1+z^{-1}} + \frac{3/4}{1-z^{-1}} + \frac{1/2\, z^{-1}}{(1-z^{-1})^2}$

$$x(n) = \mathcal{Z}^{-1}\{X(z)\} = \frac{1}{4}(-1)^n u(n) + \frac{3}{4}u(n) + \frac{1}{2}\underbrace{n\,u(n)}_{\text{ramp}}$$

Finding the Transient + Steady-State Responses of LTI Systems using the Z-Transform

Example:

$$x(n) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(n) = 0.5\,y(n-1) + x(n)$$

Assume $x(n) = 10\cos\left(\frac{\pi}{4}n\right)u(n)$

Find (i) Transient Response

(ii) Steady State

(iii) After how many samples will the Transient response vanish?

$Y(z) = H(z)X(z)$

$H(z) = ?$

$a = [1, -0.5]$, $b = 1$

$H(z) = \dfrac{1}{1 - 0.5z^{-1}}$ Stable because pole $= 0.5$

$$X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\{10\cos\left(\tfrac{\pi}{4}n\right)u(n)\}$$

$$= \frac{10\{z^2 - z\cos\omega_0\}}{z^2 - 2z\cos\omega_0 + 1} \quad \text{where } \omega_0 = \frac{\pi}{4} \text{ rad.}$$

$$\therefore Y(z) = H(z)X(z) = \frac{10\{z^2 - \frac{1}{\sqrt{2}}z\}}{z^2 - \sqrt{2}z + 1} \cdot \frac{z}{z - 0.5}$$

Find poles of $Y(z)$: $z_{p_1} = 0.5$

Roots of $z^2 - \sqrt{2}z + 1 = 0$

$\Rightarrow (z - e^{j\pi/4})(z - e^{-j\pi/4})$

$z_{p_{2,3}} = e^{\pm j\pi/4}$

Use Partial Fraction Expansion to find $y(n)$:

$$\frac{Y(z)}{z} = \frac{10z\{z - \overset{?=1}{\textcircled{z}}/\sqrt{2}\}}{(z-0.5)(z - e^{j\pi/4})(z - e^{-j\pi/4})}$$

$$z \qquad (z-0.5)(z-e^{j\pi/4})(z-e^{-j\pi/4})$$

$$= \frac{A_1}{z-0.5} + \frac{A_2}{(z-e^{j\pi/4})} + \frac{A_2^*}{(z-e^{-j\pi/4})}$$

$$A_1 = \left.\frac{10z\{z-z/\sqrt{2}\}}{(z-e^{j\pi/4})(z-e^{-j\pi/4})}\right|_{z=0.5} = -1.9$$

?

$$A_2 = \left.\frac{10z\{z-z/\sqrt{2}\}}{(z-0.5)(z-e^{-j\pi/4})}\right|_{z=e^{j\pi/4}=\frac{1}{\sqrt{2}}\{1+j\}}$$

$$= 6.78e^{-j0.5}$$

$$\therefore Y(z) = \frac{A_1}{1-0.5z^{-1}} + \frac{A_2}{1-e^{j\pi/4}z^{-1}} + \frac{A_2^*}{1-e^{-j\pi/4}z^{-1}}$$

$$y(n) = \mathcal{Z}^{-1}\{Y(z)\}$$

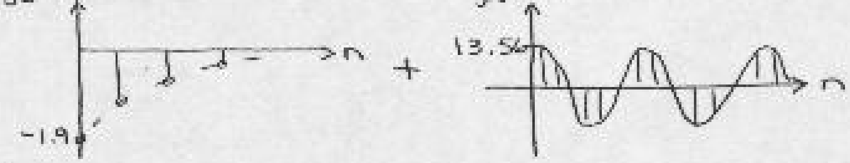
$$y(n) = -1.9(0.5)^n u(n) + \{6.78e^{-j0.5}e^{j\pi/4 n} + 6.78e^{j0.5}e^{-j\pi/4 n}\}$$

$$= -1.9(0.5)^n u(n) + 2\,\mathrm{Real}\{6.78e^{-j0.5}e^{j\pi/4 n}\}u(n)$$

$$= -1.9(0.5)^n u(n) + 2(6.78e^{j(\pi/4 n-0.5)})u(n)$$

$$= -1.9(0.5)^n u(n) + 2(6.78\cos(\tfrac{\pi}{4}n-0.5)u(n))$$

$$= \underbrace{-1.9(0.5)^n u(n)}_{\text{Transient Response}} + \underbrace{13.56\cos(\overset{\omega_0}{\tfrac{\pi}{4}}n-0.5)u(n)}_{\text{Steady State Response}}$$



for all $n \geq n_{ss}$

$$|y_{tr}(n_{ss})| \leq 0.01 \times 13.56$$

$$1.9(0.5)^{n_{ss}} \leq 0.1356$$

$$1.9(0.5)^{n_{ss}} \le 0.1356$$

$$(0.5)^{n_{ss}} = \frac{0.1356}{1.9}$$

$$n_{ss} \log_{10}(0.5) = \log_{10}\left(\frac{0.1356}{1.9}\right)$$

$$\therefore n_{ss} = \text{ceil}\left(\frac{\log\left(\frac{0.1356}{1.9}\right)}{\log(0.5)}\right)$$

Variations:

$$y_{tr}(n) = \underset{\substack{\uparrow \\ \text{will disappear} \\ \text{faster}}}{(0.5)^n u(n)} + \underset{\substack{\uparrow \\ \text{will dominate}}}{(0.8)^n u(n)}$$

Steady-state verification:

$$y_{ss}(n) = 13.56 \cos(\omega_0 n - 0.5) u(n)$$

$$\omega_0 = \pi/4 \text{ rad.}$$

Verify this result is correct:

Frequency Response $H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$

$$\Rightarrow H(e^{j\omega}) = \frac{1}{1-0.5e^{-j\omega}}$$

$$= \frac{1}{1-0.5e^{-j\pi/4}} = |H(e^{j\omega_0})| \measuredangle H(e^{j\omega_0})$$

?

$$y_{ss}(n) = 10 \underset{\substack{\downarrow \\ 1.356}}{|H(e^{j\omega_0})|} \cos(n\omega_0 + \underset{\substack{\downarrow \\ -0.5 \text{ rad.}}}{\measuredangle H(e^{j\omega_0})})$$

Example:

Find the impulse response of $H(z) = \dfrac{1-3z^{-1}}{1-0.5z^{-1}}$

$$h(n) = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{z-3}{z-0.5}\right\}$$

$$\frac{H(z)}{z} = \frac{1}{z}\left\{\frac{z-3}{z-0.5}\right\} = \frac{A}{z} + \frac{B}{(z-0.5)}$$

$$A = 6, \quad B = -5$$

$$\frac{H(z)}{z} = \frac{6}{z} - \frac{5}{z - 0.5}$$

$$H(z) = 6 - \frac{5}{1 - 0.5z^{-1}}$$

$$\therefore h(n) = 6\delta(n) - 5(0.5)^n u(n)$$

Find the step response

$$Y(z) = X(z)H(z)$$

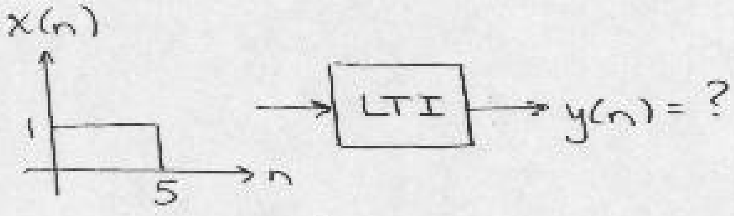
$$X(z) = \mathcal{Z}\{u(n)\} = \frac{1}{1 - z^{-1}}$$

$$Y(z) = \frac{1}{1 - z^{-1}} \cdot \frac{1 - 3z^{-1}}{1 - 0.5z^{-1}}$$

Use Partial Fraction Expansion to find step response

Example:

Find response to x(n)



$$x(n) = u(n) - u(n-6)$$

$$\downarrow{\scriptstyle \text{response}} \qquad \downarrow{\scriptstyle \text{response}}$$

$$s(n) \; - \; s(n-6)$$

↳ This response was found in above example.

2/22

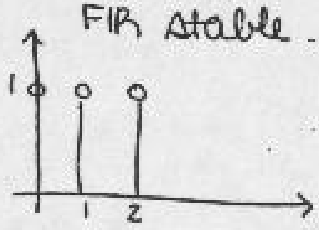
Exam One

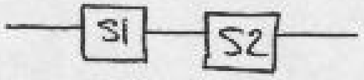
① S1: $y(n) = x(n) + x(n-1) + x(n-2)$

S2: $h(n) = \delta(n) - \delta(n-1)$

S1: $h(n) = \delta(n) + \delta(n-1) + \delta(n-2)$

FIR Stable





$h(n) = h_1(n) * h_2(n)$

$x(n) * \delta(n) = x(n)$

$x(n) * \delta(n-n_0) = x(n-n_0)$

$$H(e^{j\omega}) = H_1(e^{j\omega}) H_2(e^{j\omega})$$
$$= \{1 + e^{-j\omega} + e^{-j2\omega}\}\{1 - e^{-j\omega}\}$$

$$= \{1 + e^{-j\omega} + e^{-j2\omega}\}\{1 - e^{-j\omega}\}$$
$$= 1 - e^{-j3\omega}$$

$$FT^{-1}\{H(e^{j\omega})\} = h(n) = \delta(n) - \delta(n-3)$$

note:
$e^{j0} = 1$
$e^{j\pi/2} = j$
$e^{j\pi} = -1$
$e^{j2\pi} = 1$

$$H(e^{j0}) = 0$$
$$H(e^{j\pi/2}) = 1 - e^{-j3\pi/2} = 1 - j$$
$$H(e^{j\pi}) = 1 - e^{j3\pi} = 1 - (-1) = 2$$

HPF

5.5

② $y(n) - 0.5y(n-1) = 0.5x(n) + x(n-1)$

IIR

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{-0.5 + e^{-j\omega}}{1 - 0.5e^{-j\omega}}$$
$$= \frac{-0.5 + e^{-j\omega}}{e^{-j\omega}\{-0.5 + e^{j\omega}\}}$$

$$|H(e^{j\omega})| = \frac{|-0.5 + e^{-j\omega}|}{|e^{-j\omega}||-0.5 + e^{j\omega}|} = 1 \quad \text{Allpass}$$

pole: $z_p = 0.5$ stable

Let $x(n) = u(n) \Longrightarrow y(n) = s(n)$

$s(n) = 0.5s(n-1) - 0.5u(n) + u(n-1)$

$s(0) = 0 - 0.5 + 0 = -0.5$

$s(1) =$

$s(2) =$

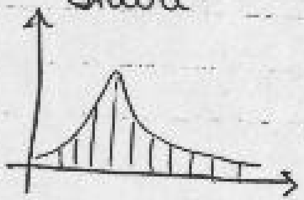
$S(2) =$

$\vdots$

$h(1) = S(1) - S(0)$ (first diff. of unit-step response)

③ $h(n) = 0.5(0.8)^n u(n) - 0.5(0.4)^n u(n)$

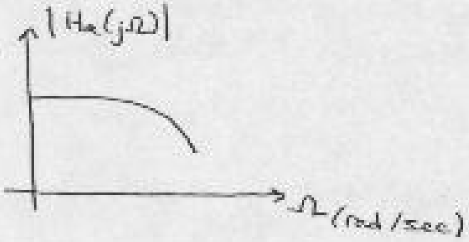
stable

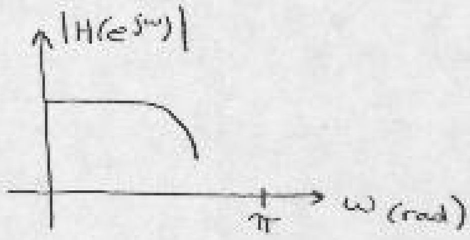


Stable: $\sum_0^\infty 0.8^n < \infty$

2-24

Design of IIR Filters

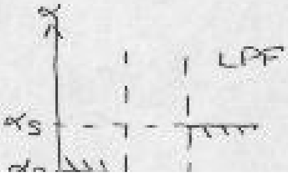


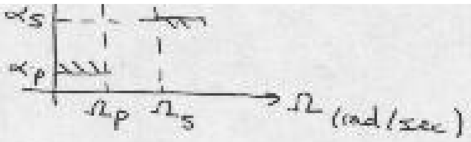


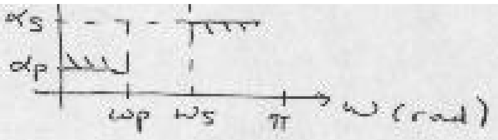
$$H(z) = H_a(s)\Big|_{s = \frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})}}$$

$$H(e^{j\omega}) = H_a(j\Omega)$$
$$\Omega = \frac{2}{T}\tan(\omega/2)$$

Example:



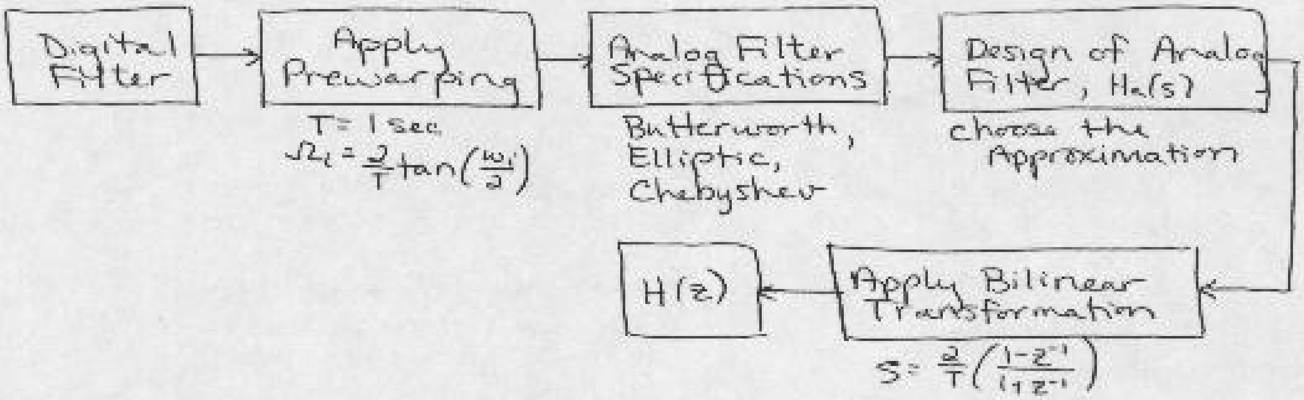




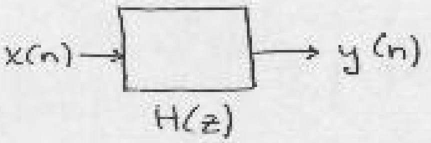
$\alpha = -20\log|H_a(j\Omega)|$

$\alpha = -20\log|H(e^{j\omega})|$ (dB)

$$\Omega_p = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right)$$
$$\Omega_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right)$$

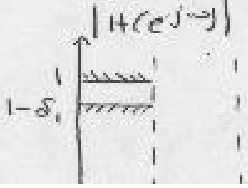


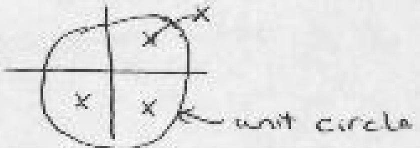
Design of Digital Filters

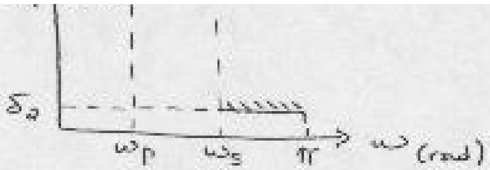


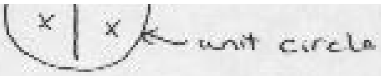
$y(n) = y(n-1) \ldots$ Diff. Eqn

① Magnitude Response, $|H(e^{j\omega})|$





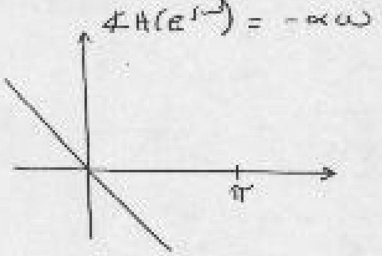




IIR

-Check for Stability
$H(z) \rightarrow$ find poles $\rightarrow$ inside unit circle

② Phase Response, $\measuredangle H(e^{j\omega})$

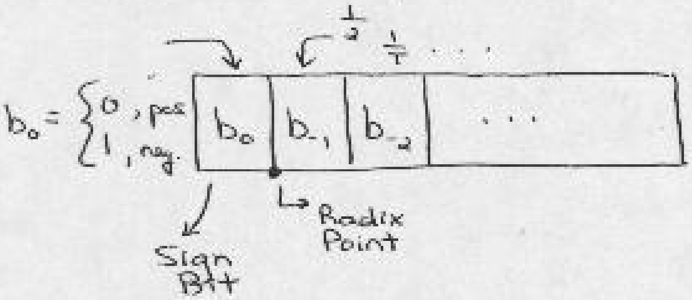


-Meeting both mag. Response & a Linear Phase
FIR

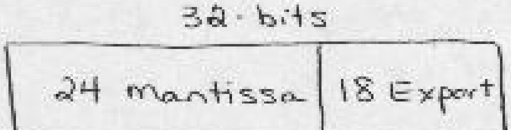
$y(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \cdots$
Non-recursive

Order: $n_{FIR} >> n_{IIR}$

Fixed Point Rep.



Floating Point Rep.



24 mantissa	18 Export

$$mba = mantissa \times 2^{Export}$$

Example:

$$y(n) = x(n) + x(n-1) + x(n-2)$$

$$y(0),\ y(100),\ y(200)$$

Methods to Design IIR Filters

Design an Analog Filter

$$\boxed{H_a(s)} \xrightarrow{\text{Apply a Transformation}} \text{to obtain } H(z)$$

- * Butterworth
- * Chebyshev I
- * Chebyshev II
- * Elliptic

$$H(z) = H_a(s)\Big|_{s=f(z)}$$

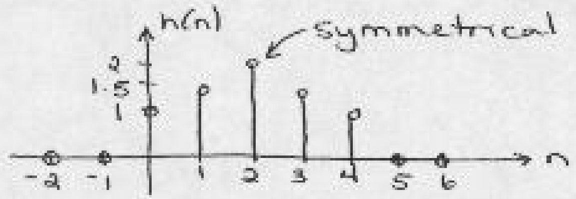
↳ Transformation (Bilinear Trans.)

Optimization methods:

$$z = f(z_1, z_2)$$

FIR Filters with Linear Phase (Constant Group Delay)

Example



In order to make linear phase, make symmetrical.

$$H(e^{j\omega}) = FT\{h(n)\} \qquad h(0) = h(4)$$

$$= \sum_{0}^{4} h(n)e^{-j\omega n} \qquad h(1) = h(3)$$

$$= h(0) + h(1)e^{-j\omega} + h(2)e^{-j2\omega}$$

$$+ h(4)e^{-j4\omega} + h(3)e^{-j3\omega}$$

$$= h(0)\{1+e^{-j4\omega}\} + h(1)\{e^{-j\omega}+e^{-j3\omega}\} + h(2)e^{-j2\omega}$$

$$= e^{-j2\omega}\{h(0)\{e^{j2\omega}+e^{-j2\omega}\} + h(1)\{e^{j\omega}+e^{-j\omega}\} + h(2)\}$$

$$H(e^{j\omega}) = e^{-j2\omega}\underbrace{\{2h(0)\cos 2\omega + 2h(1)\cos\omega + h(2)\}}_{\text{Real Number} = A(\omega)}$$

$$= e^{-j2\omega}A(\omega)$$

Phase Response:

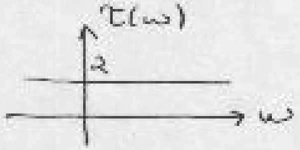
$$\measuredangle H(e^{j\omega}) = \measuredangle e^{-j2\omega} + \measuredangle A(\omega)$$

$$= -2\omega + \begin{cases} 0 & \text{if } A(\omega) \geq 0 \\ \pi & \text{if } A(\omega) < 0 \end{cases}$$

<u>Generalized Linear Phase</u>

Group Delay:

$$\tau(\omega) = \frac{d}{d\omega}\left(\measuredangle H(e^{j\omega})\right) = 2 \text{ samples}$$

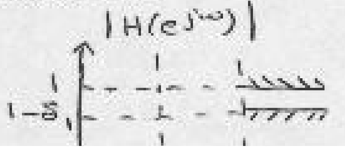


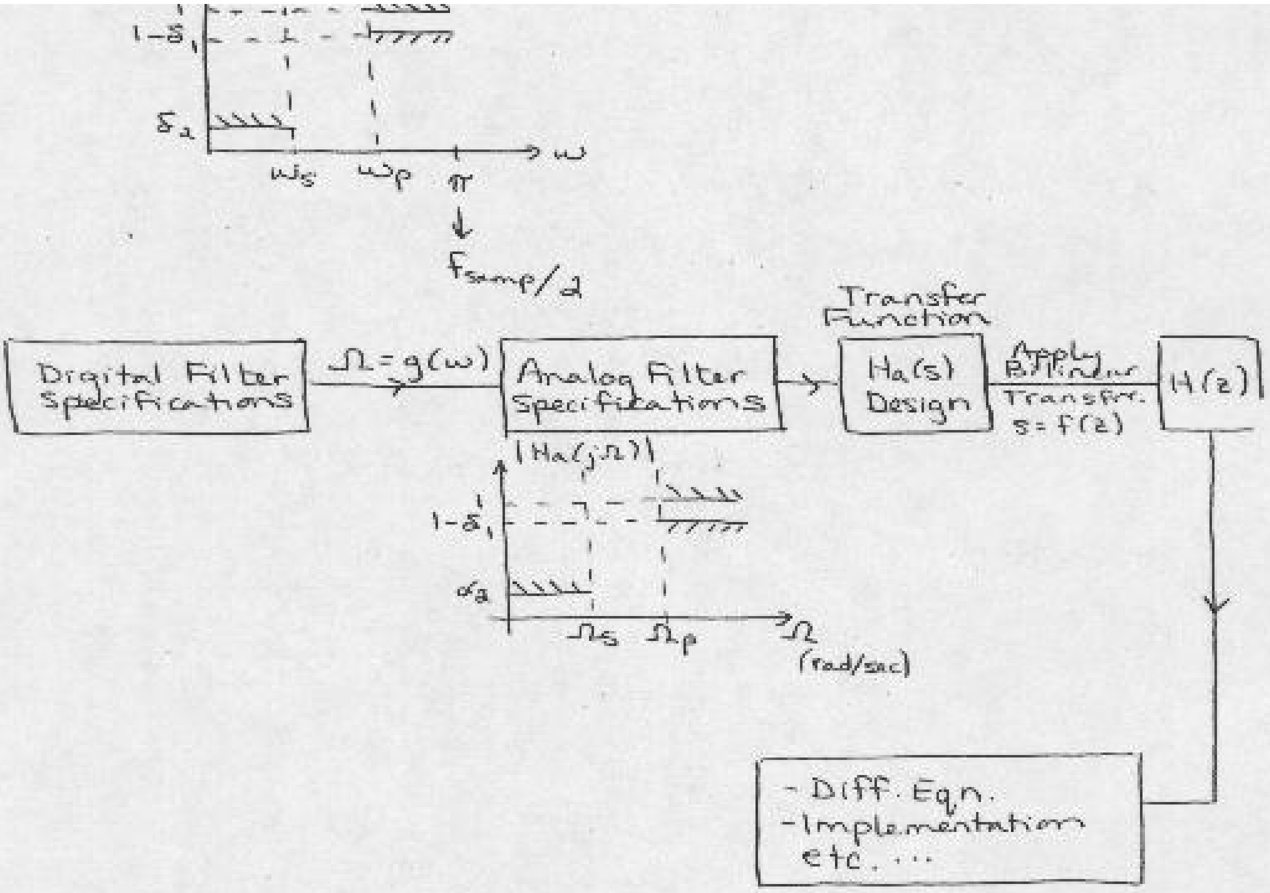
Design of IIR Filters

method:

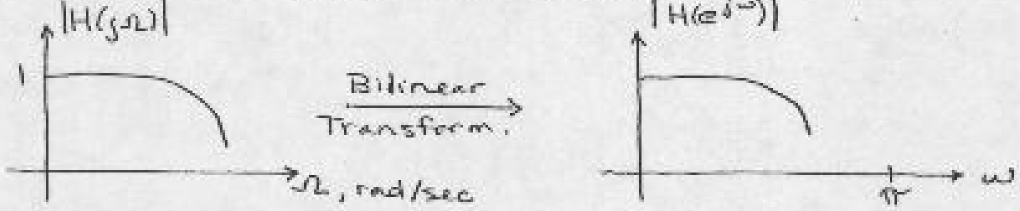
Apply the bilinear transformation to analog filter

Procedure:





The Bilinear Transformation



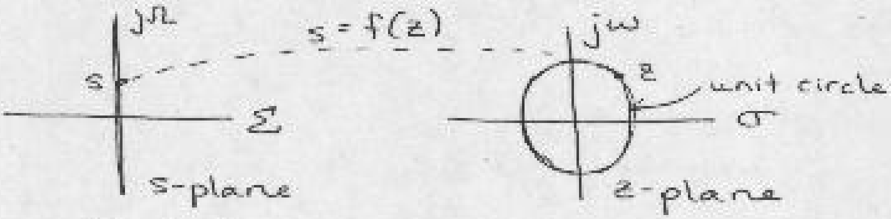
$H_a(s) = 1$ Transfer Function $H(z) = H_a(s)\big|$ $= \frac{1}{1+f(z)}$

$H_a(s) = \frac{1}{s+1}$ Transfer Function $H(z) = H_a(s)\big|_{s=f(z)} = \frac{1}{1+f(z)}$

$H_a(j\Omega) = \frac{1}{1+j\Omega}$ freq. Response $H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \frac{1}{1+f(e^{j\omega})}$

In order to make both freq. resp. the same:

$$s = f(z)\big|_{z=e^{j\omega}} = j\Omega$$



$s = \Sigma + j\Omega$ $z = \sigma + j\omega$

$$z = f^{-1}(s)$$

- Any s on imag. axis must be mapped onto unit circle in z-plane
- Any z on unit circle must be mapped onto imag. axis in s-plane

$H_a(s) \longrightarrow$ stable $\longrightarrow$ poles of $H_a(s)$ (LHS of s-plane)

- Any poles in LHS of s-plane must be mapped inside unit circle in z-plane
- Any poles inside unit circle must be mapped into LHS of s-plane

---

Requirements of the Bilinear Transformation

1) The imag. axis of the s-plane will be transformed onto the unit circle in the z-plane

the z-plane

$$H(e^{j\omega}) = H_a(j\Omega)$$

2) The LHS of the s-plane must be mapped onto the area inside the unit circle in the z-plane

The Bilinear Transformation

$$s = f(z) = \frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})}$$ (T is a constant, not period)

Conversely: $$z = \frac{1+\frac{T}{2}s}{1-\frac{T}{2}s}$$

Take a point on the unit circle in the z-plane; Thus $z = e^{j\omega}$

Therefore, the corresponding point in the s-plane will be

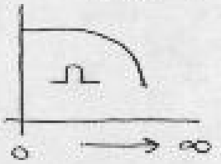
$$s = \frac{2}{T}\frac{(1-e^{-j\omega})}{(1+e^{-j\omega})} = \frac{2}{T}\frac{e^{-j\omega/2}}{e^{-j\omega/2}}\frac{\{e^{j\omega/2}-e^{-j\omega/2}\}}{\{e^{j\omega/2}+e^{-j\omega/2}\}}$$

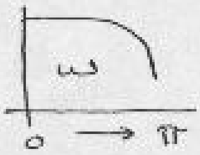
$$= \frac{2}{T}\frac{j\sin \omega/2}{\cos \omega/2}$$

$$s = \frac{j2}{T}\tan(\omega/2) = j\Omega$$

$$\Rightarrow \boxed{\Omega = \frac{2}{T}\tan(\omega/2)}$$

$$H_a(j\Omega) = H(e^{j\omega})$$





$$\begin{array}{ccc} \Omega & & \omega \\ 0 & \longrightarrow & 0 \\ \infty & \longrightarrow & \pi \end{array}$$

Stability

Take a point in the s-plane located in the LHS of the s-plane

$$S = \Sigma + j\Omega \quad , \text{ where } \Sigma < 0$$

Therefore,

$$z = \frac{1 + T/2 \{ \Sigma + j\Omega \}}{1 - T/2 \{ \Sigma + j\Omega \}}$$

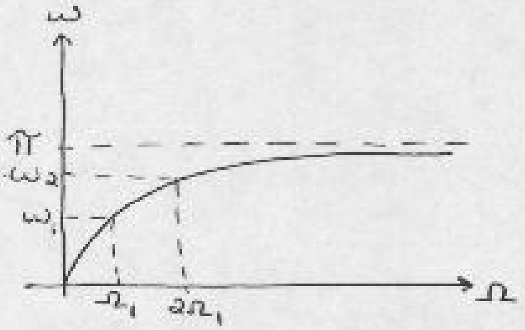
$$= \frac{\overbrace{1 + \frac{T\Sigma}{2}}^{\alpha} + \frac{jT}{2}\Omega}{\underbrace{1 - \frac{T\Sigma}{2}}_{\beta} - \frac{jT}{2}\Omega}$$

$$= \frac{\alpha + \frac{jT}{2}\Omega}{\beta - \frac{jT}{2}\Omega}$$

$$\beta > \alpha \qquad \Rightarrow |z| = \frac{\sqrt{\alpha^2 + (\frac{T}{2}\Omega)^2}}{\sqrt{\beta^2 + (\frac{T}{2}\Omega)^2}} < 1$$

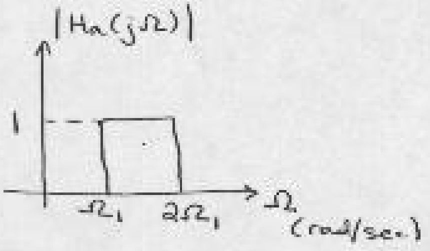
$\therefore$ inside unit circle in z-plane

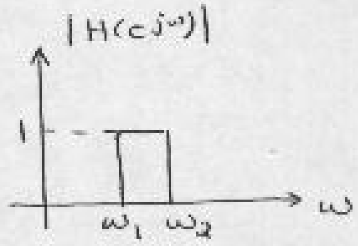
$$\boxed{\begin{aligned} H_a(j\Omega) &= H(e^{j\omega}) \\ \Omega &= \frac{2}{T} \tan\left(\frac{\omega}{2}\right) \\ \omega &= 2 \tan^{-1}\left(\frac{\Omega T}{2}\right) \end{aligned}}$$



As $\Omega \to \infty$, $\omega \to \pi$

$$H_a(j0) = H(e^{j0})$$
$$H_a(j\infty) = H(e^{j\pi})$$





$$H(e^{j\omega}) = \underline{FT}\{0.5(0.8)^n u(n) - 0.5(0.4^n u(n)\}$$

$$= \frac{0.5}{1-0.8e^{-j\omega}} - \frac{0.5}{1-0.4e^{-j\omega}}$$

$$= \frac{0.5(1-0.4e^{-j\omega} - 1 + 0.8e^{-j\omega})}{(1-0.8e^{-j\omega})(1-0.4e^{-j\omega})}$$

$$= \frac{0.2e^{-j\omega}}{1-1.2e^{-j\omega}+0.32e^{-j2\omega}}$$

$$b = [0, 0.2]$$

$$a = [1, -1.2, 0.32]$$

$$y(n) = 1.2y(n-1) - 0.32y(n-2) + 0.2x(n-1)$$

$$x(n) = \cos(\overset{y}{\tfrac{3\pi}{4}}n)\sin(\overset{x}{\tfrac{\pi}{2}}n)u(n)$$

$$\sin(x)\cos(y) = \frac{1}{2}\{\sin(x+y) + \sin(x-y)\}$$

$$= \frac{1}{2}\{\sin x\cos y + \cos x \sin y\}$$

$$\therefore x(n) = \frac{1}{2}\{\sin((\tfrac{2\pi}{4} + \tfrac{3\pi}{4})n) - \sin((\tfrac{3\pi}{4} - \tfrac{2\pi}{4})n)\}u(n)$$

$$= \frac{1}{2}(\sin \pi n - \frac{1}{2}\sin(\frac{\pi}{2}n))u(n)$$

$$= \frac{1}{2}\cos(\frac{\pi}{2}n + \pi/2)u(n)$$

$$y_{ss}(n) = \frac{1}{2}|H(e^{j\pi/2})|\cos(\frac{\pi}{2}n + \frac{\pi}{2}) + \measuredangle H(e^{j\pi/2})$$

Example:

$\alpha_p = 3$ dB & $\omega_p = 0.5\pi$ rad. Digital LPF

$\alpha_s = 15$ dB & $\omega_s = 0.75\pi$ rad. Requirements.

Take T=1

$$\Omega_p = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right) = 2\tan\left(\frac{0.5\pi}{2}\right) = 2 \text{ rad/sec}$$

$$\Omega_s = \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right) = 2\tan\left(\frac{0.75\pi}{2}\right) = 4.825 \text{ rad/sec}$$

Design the analog filter:

Take the Butterworth approximation (2nd order)

$$H_a(s) = \frac{1}{(s/2)^2+\sqrt{2}(s/2)+1}$$

n=2

$\Omega_c = 2$ rad/sec

$$= \frac{4}{s^2+2\sqrt{2}s+4}$$

Apply Bilinear Transformation:

$$H(z) = H_a(s)\Big|_{s=\frac{2(1-z^{-1})}{T(1+z^{-1})}} = \frac{4}{\left(2\frac{(1-z^{-1})}{(1+z^{-1})}\right)^2+2\sqrt{2}\left(2\frac{(1-z^{-1})}{(1+z^{-1})}\right)+4}$$

$$= \frac{b_0+b_1z^{-1}+b_2z^{-2}}{a_0+a_1z^{-1}+a_2z^{-2}}$$

Same example using matlab:

```
% Design analog filter
WP = 2;                                % Passband freq
WS = 4.825;                            % Stopband freq
% Butterworth Approx.
% n=2 & WC = cutoff
[n, WC] = buttord(WP, WS, aP, aS, 's');
                                          analog filter
{  aP = 3;   % Passband Attenuation
{  aS = 15;  % Stopband Attenuation

[banalog, aanalog] = butter (n, WC, 's');
% [0 0 4.2073]        [1 2.9008 4.2073]
```

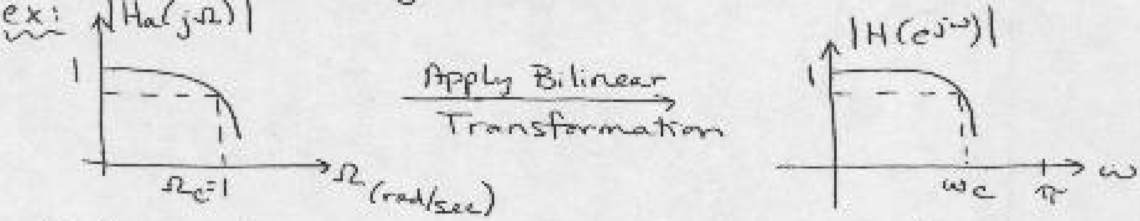
% $H_a(s) = \dfrac{4.2073}{s^2+2.9008s+4.2073}$

$$\% \quad H_a(s) = \frac{4.2073}{s^2 + 2.9008s + 4.2073}$$

```
% Apply Bilinear Transformation
T= 1;
[b, a] = bilinear(banalog, aanalog, 1/T);
%   ↖ Digital Filter Transfer Function Coeff.
% Check the freq. response
```

First Order Digital Filter

ex:



$$H_a(s) = \frac{1}{s+1}$$

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})}}$$

$$= \frac{1}{\frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})} + 1}$$

$$= \frac{(1+z^{-1})}{\frac{2}{T}(1-z^{-1}) + (1+z^{-1})}$$

$$= \frac{(1+z^{-1})}{(\frac{2}{T}+1) - (\frac{2}{T}-1)z^{-1}}$$

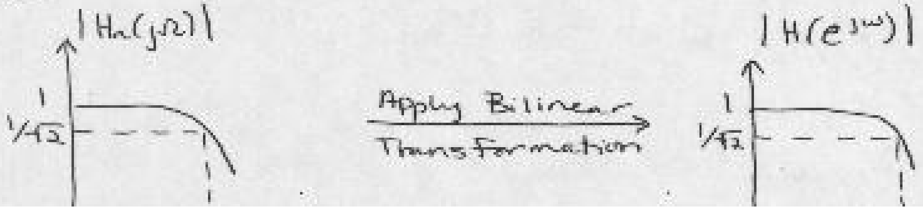
$$\omega_c = 2\tan^{-1}(\Omega_c T/2)$$

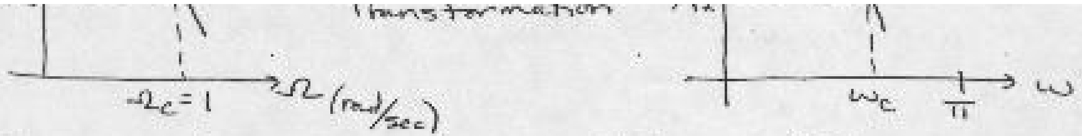
$$\Omega_c = 1$$

$$\omega_c = 2\tan^{-1}(T/2)$$

Given ω_c, find T, + plug into $H(z)$

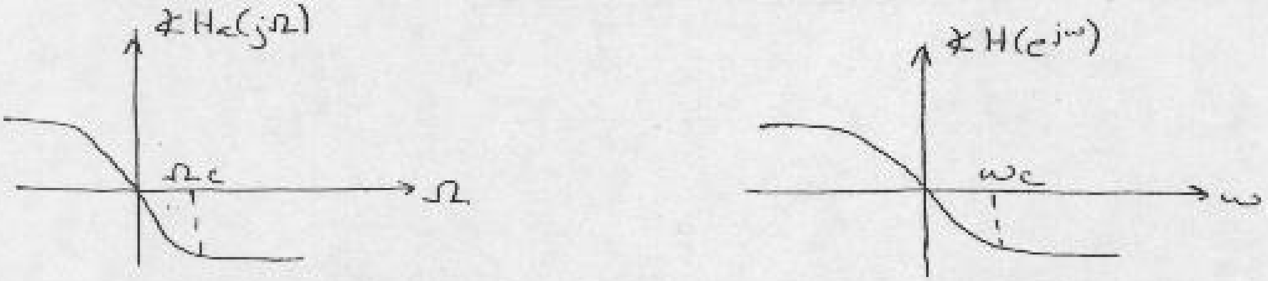
ex:





$H_a(s) = \frac{1}{}$ $H(z) = H_a(s)\Big|$

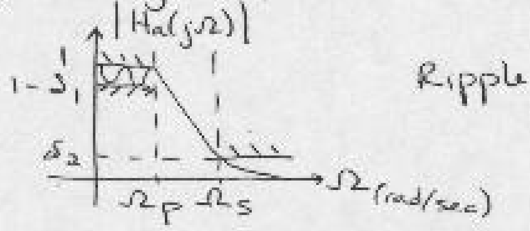
$H(e^{j\omega}) = H_a(j\Omega)$



Analog Filter Design

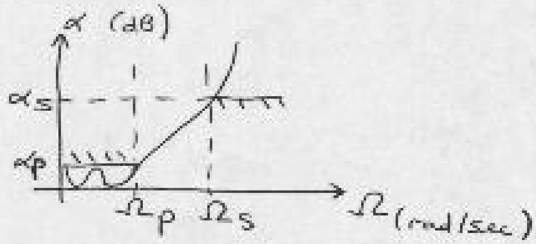
Chebyshev Approximation

1) Chebyshev I



Ripple in the Passband

All pole Transfer Function
(No Zeros)



```
[n, WN] = cheb1ord(WP, WS, aP, aS, 's');
```

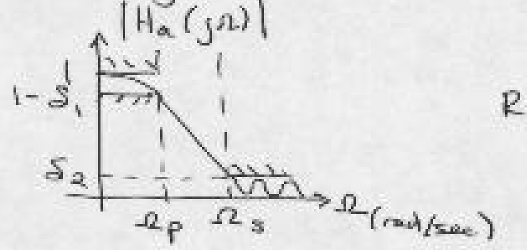
order   Ripple Bandwidth

[num, den] = cheby1 (n, aP, WN, 'filtertype', 's');

↳ 'high' ⇒ HPF

'stop' ⇒ Notch

2) Chebyshev II



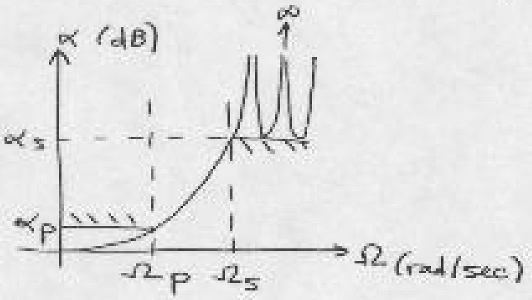
Ripple in the Stopband

Poles & Zeros

↳ on imag. axis

[n, WN] = cheb2ord (WP, WS, aP, aS, 's');

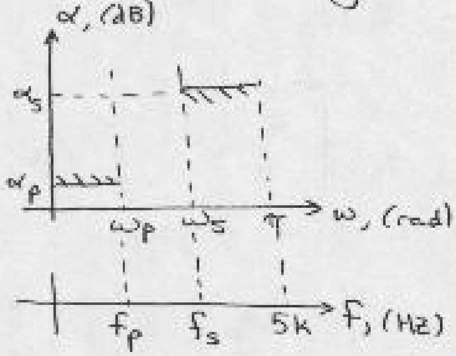
[num, den] = cheby2 (n, aS, WN, 'filtertype', 's');



75

2-29

IIR Filter Design

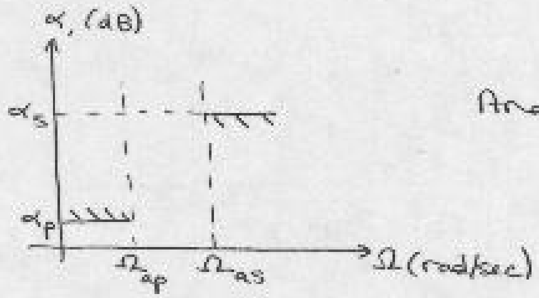


1. Butterworth
2. Chebyshev
   (i) Type #1
      - Ripple in Passband
   (ii) Type #2
      - Ripple in Stopband
3. Elliptic
      - Ripple in Pass + Stop Ban

$$\omega_p = \frac{2\pi f_p}{f_{samp}}$$

$f_{samp} = 10$ kHz

To Analog Filter →



Analog Filter

$T=1 \Rightarrow \Omega_{ap} = \frac{2}{T}\tan(\omega_p/2) = 2\tan(\omega_p/2)$

$\Rightarrow \Omega_{as} = 2\tan(\omega_s/2)$

Design $H_a(s)$:

Choose Elliptic Approximation

Choose Elliptic Approximation

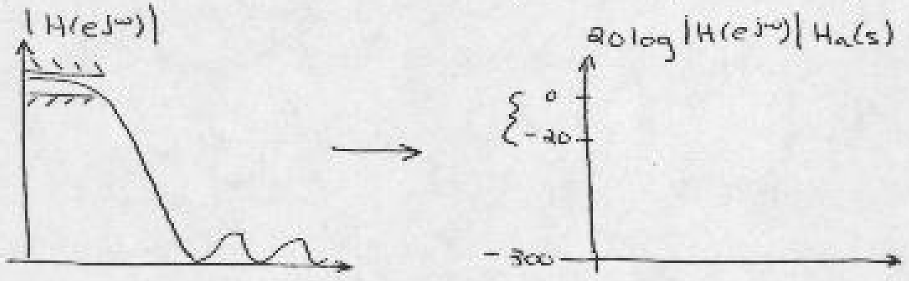
MATLAB:

```
[N, WN] = ellipord (Ωap, Ωas, αp, αs, 's');
[numa, dena] = ellip (N, αp, αs, WN, 'filter
                                     type', 's');
    % Only need 'filter type' for
      HighPass ('high') and Notch ('stop')
```

% $H_a(s) = \dfrac{b_0' s^N + b_1' s^{N-1} + \cdots}{a_0' s^N + \cdots}$

```
numa = [b0' b1' ...];
dena = [a0' a1' ...];
```

```
% Apply the bilinear transformation
[b, a] = bilinear (numa, dena, 1/T);
```

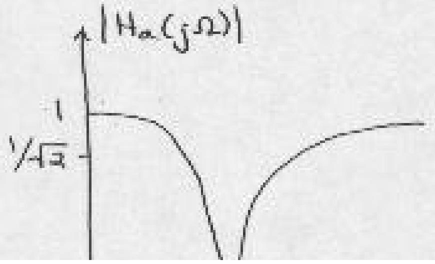


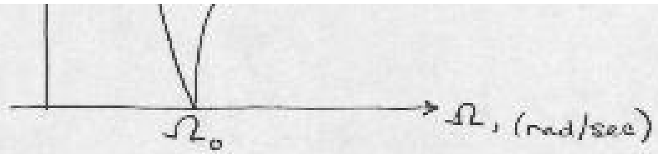
* Use axis command to limit y axis.

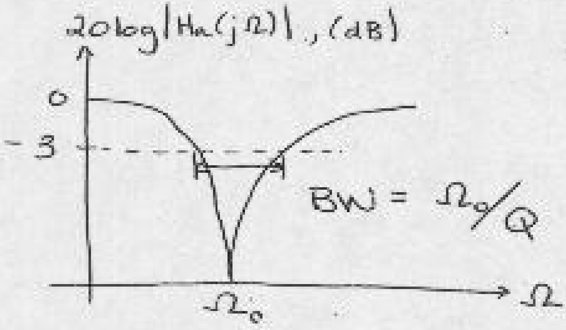
Consider a 2nd order analog notch filter:

$$H_a(s) = \frac{s^2 + \Omega_0^2}{s^2 + \frac{\Omega_0}{Q}s + \Omega_0^2}$$

$$|H_a(j\Omega)| = \frac{|\Omega_0^2 - \Omega^2|}{\sqrt{(\Omega_0^2 - \Omega^2) + \left(\frac{\Omega_0 \Omega}{Q}\right)^2}}$$







$$H_a(s) = \frac{s^2+\Omega_o^2}{s^2+BWs+\Omega_o^2}$$

Find Digital Counterpart by Applying Bilinear Transformation:

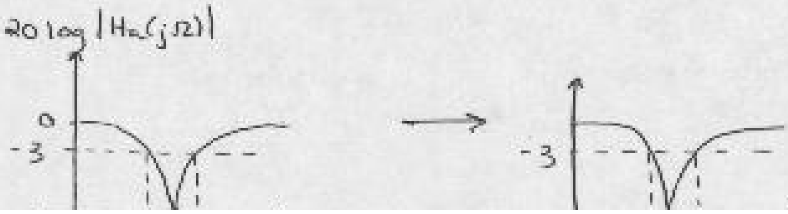
T = 2 (Arbitrary)

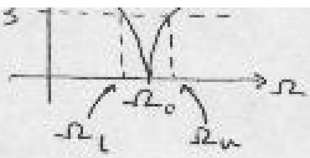
$$\therefore H(z) = H_a(s)\Big|_{s=\frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})}} = \frac{(1-z^{-1})}{(1+z^{-1})}$$

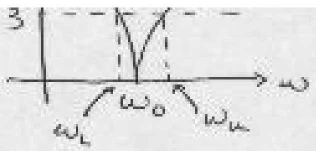
$$H(z) = \frac{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + \Omega_o^2}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + BW\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + \Omega_o^2}$$

$$H(z) = \frac{(1+\Omega_o^2) - 2(1-\Omega_o^2)z^{-1} + (1+\Omega_o^2)z^{-2}}{(1+\Omega_o^2+BW) - 2(1-\Omega_o^2)z^{-1} + (1+\Omega_o^2-BW)z^{-2}}$$

2nd order

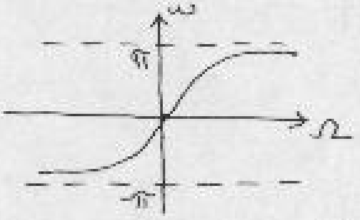






$$\omega = 2\tan^{-1}(\Omega T/2)$$

↳ digital frequency

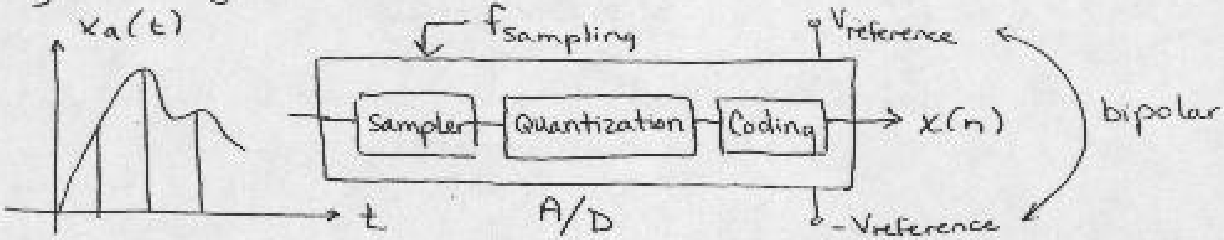


$$H_a(j\Omega) = H(e^{j\omega})$$

$$\omega_0 = 2\tan^{-1}(\Omega_0)$$

$$\omega_L = 2\tan^{-1}(\Omega_L) \qquad \Delta\omega = 2\tan^{-1}(\Omega_u) - 2\tan^{-1}(\Omega_L)$$

Analog to Digital Converters



n bits: number of bits

Quantizer Model

$$V_{swing} = 2V_{reference}$$

n_b = number of bits

Example: $n_b = 3$ bits

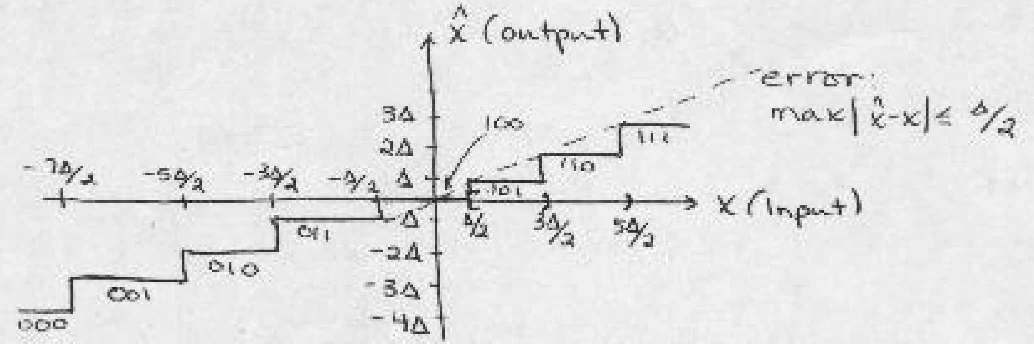
$$x \rightarrow \boxed{\text{Quantizer}} \rightarrow \hat{x} = Q(x) = x + \text{error}$$

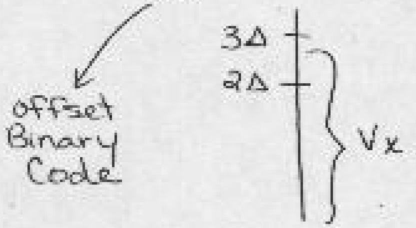
↳ Quantization Noise

Number of levels $= 2^{n_b} = 2^3 = 8$ levels.

Number of levels = $2^{n_b} = 2^3 = 8$ levels.

Δ = Step size = $\frac{2V_{reference}}{2^{n_b}} = 2^{(1-n_b)} \cdot V_{reference}$





$\Delta * \text{round}\left(V_x/\Delta\right)$
$\hookrightarrow$ multiplication

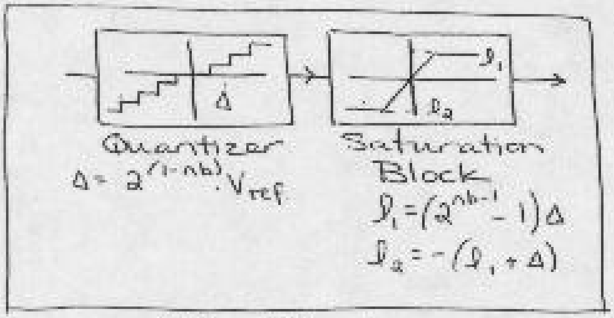
$2 \rightarrow 3$: $\Delta * \text{round}(V_x/\Delta) = 3\Delta$

3-2

Offset Binary Code → 2's Complement Code

000 → 100
001 → 101
010 → 110
011 → 111
100 → 000
101 → 001
110 → 010
111 → 011

Model in Simulink



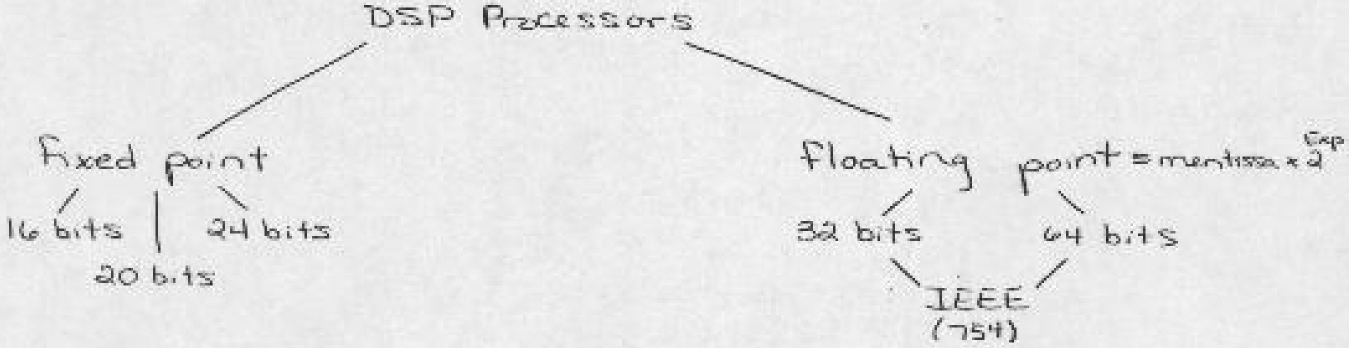
A/D Block

A/D Block

Matlab A/D Function

```
function y = adc(x, n, Vref);      % x: input , n: # bits
xm = max(abs(x));
Xnorm = x(xm);                     % normalization
delta = 2^(1-n);
l1 = 1 - delta;
l2 = -1;
% Approximation
lx = Length(x);
xq = zeros(size(n));
for i = 1: lx
    xq(i) = delta * round(x(i)/delta);
    if xq(i) >= l1
        xq(i) = l1;
    elseif xq(i) < l2
        xq(i) = l2;
    end
end
```

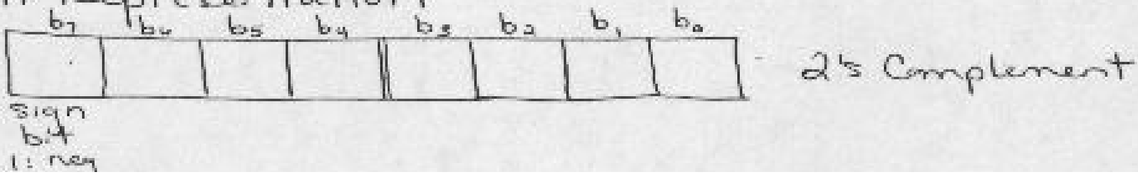
Binary Numeric Representation



Fixed Point Representation (-1 → 1)

- Integers

8-bit Representation



sign
bit
1: neg
0: pos.

$$\text{Value} = b_0 2^0 + b_1 2 + b_2 2^2 + \cdots + b_6 2^6 - b_7 2^7$$

Example: 0101 0011 (Positive)

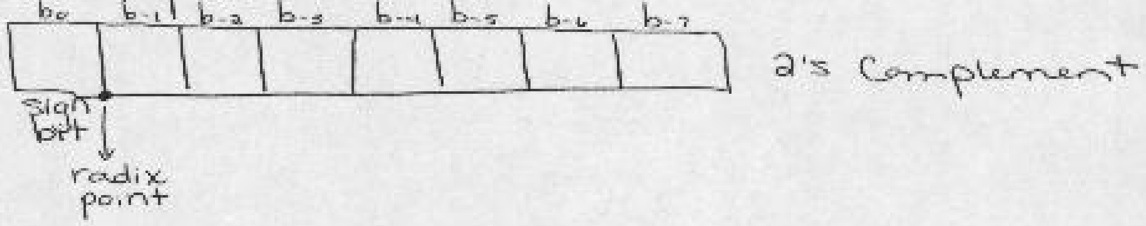
Value = 1+2+16+64 = 83

Example: 1010 1100 (Negative)

Value = 4+8+32−128= −84

- Fractional Representation

8-bit Representation



$$\text{Value} = -b_0 + b_{-1} 2^{-1} + b_{-2} 2^{-2} + \cdots + b_{-7} 2^{-7}$$

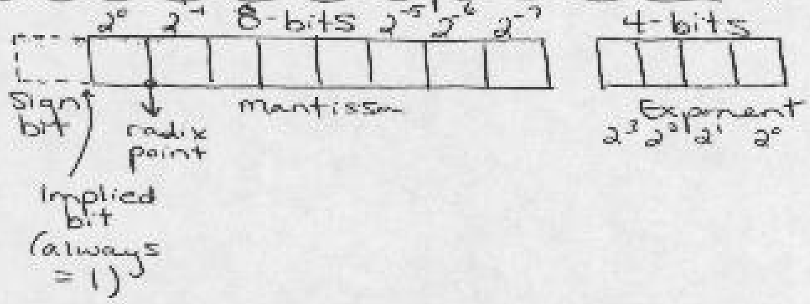
Example: 0101 0000 (Positive)

$$\text{Value} = \tfrac{1}{2} + 0 + \tfrac{1}{8} + 0 + 0 + 0 + 0 + 0 = 0.625$$

Example: 1010 1000 (Negative)

$$\text{Value} = -1 + \tfrac{1}{4} + \tfrac{1}{16} = -0.6875$$

Floating Point Representation (−2 → 2)



Binary Numeric Representation

① Fixed Point Representation (16, 20, 24 bits)

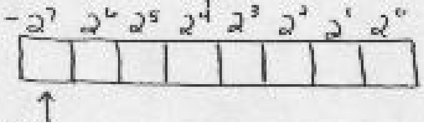
② Floating Point (32, 64 bits, IEEE754 format)

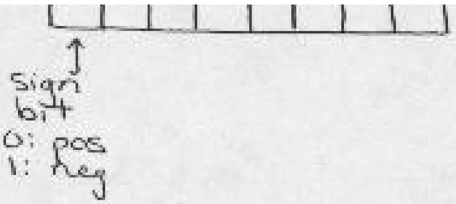
$$\text{number} = \text{mantissa} \times 2^{\text{exp}}$$

Fixed Point Numbers

a) Integers

(Two's Complement)





ex: 01010011

number = $2^6 + 2^4 + 2^1 + 2^0 = 83$

ex: 10101100

number = $-2^7 + 2^5 + 2^3 + 2^2 = -128 + 32 + 8 + 4 = -84$

b) Fractional

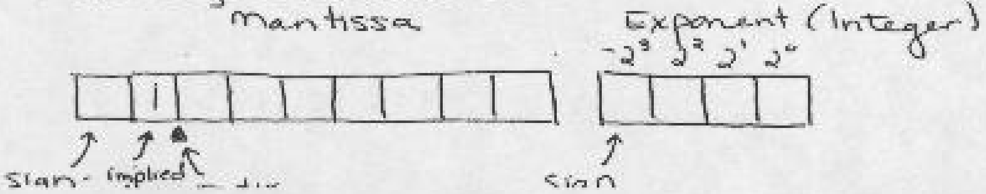
ex: 01010000

number = $(0 \times -1) + 2^{-1} + 2^{-3} = 0.625$

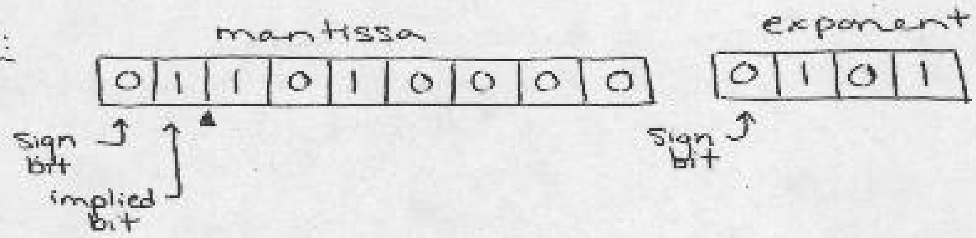
ex: 10101000

number = $(1 \times -1) + 2^{-2} + 2^{-4} = -0.6875$

Floating Point Numbers



ex:



$\text{mantissa} = \begin{cases} -2 \\ 2 \end{cases}$

$\text{mantissa} = 1 + 2^{-1} + 2^{-3} = 1.625$

mantissa = $1 + 2^{-1} + 2^{-3} = 1.625$

exponent = $2^{0} + 2^{2} = 5$

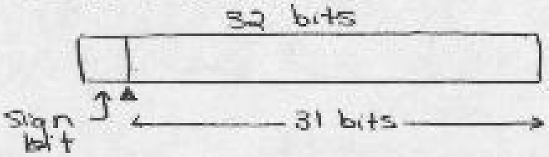
∴ number = $1.625 \times 2^{5} = 52$

Dynamic Range Comparison of Fixed Point and Floating Point Representations

ex:

Consider a 32-bit Case

* Fixed Point Fractional Representation



$n_{smallest} \equiv \boxed{0}\boxed{0}\cdots\cdots\boxed{0}\boxed{1} = 2^{-31}$

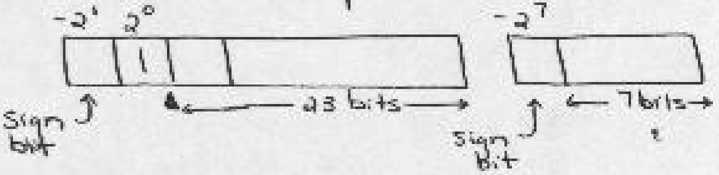
$n_{largest} \equiv \boxed{0}\boxed{1}\cdots\cdots\boxed{1}\boxed{1} = 1 - 2^{-31}$

$$\text{Ratio} = \frac{n_{largest}}{n_{smallest}} = \frac{1-2^{-31}}{2^{-31}} \Rightarrow \text{in dB}: 20\log(2^{31}-1) = 187 \text{ dB}$$

* Floating Point Representation

24 bits for mantissa (one implied bit is added)

8 bits for exponent



$n_{smallest}$: Exponent $\equiv \boxed{1}\boxed{0}\cdots\boxed{0}\boxed{1} = -2^{7} + 1 = -127$

mantissa $\equiv \boxed{0}\boxed{1}\boxed{0}\cdots\cdots\boxed{0} = 1$

$n_{smallest} = 2^{-127}$

$n_{largest}$: Exponent $\equiv \boxed{0}\boxed{1}\cdots\boxed{1}\boxed{1} = 128 - 1 = 127$

$n_{largest}$: exponent $= \boxed{0 \mid 1 \mid \cdots \mid 1 \mid 1} = 128-1 = 127$

Mantissa $\equiv \boxed{0 \mid 1 \mid \cdots\cdots \mid 1} \approx 2$

$$n_{largest} = 2 \times 2^{127} = 2^{128}$$

$$\text{Ratio} = \frac{n_{largest}}{n_{smallest}} = \frac{2^{128}}{2^{-127}} \Rightarrow \text{in dB : } 1535 \text{ dB}$$

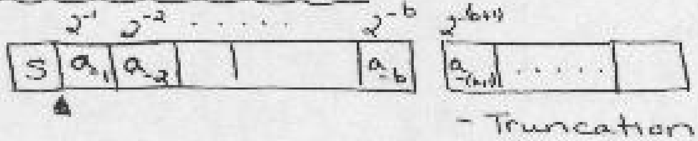
Finite Word Length Effects

$$a_0 y(n) + a_1 y(n-1) + \cdots + a_N y(n-N)$$

$$\underset{=1}{\underset{\downarrow}{a_0}} y(n) + a_1 y(n-1) + \cdots + a_N y(n-N)$$
$$= b_0 x(n) + b_1 x(n-1) + \cdots + b_N x(n-N)$$

8 bits Stability Freq. Resp. Spec.
32 bits

Fractional Numbers



using (b+1) bits.

- Truncation (Error)
- Rounding
  If $a_{-(b+1)} = 1$ then make
  $a_{-b} = 1$

Matlab Code:

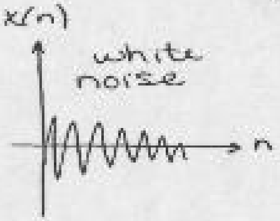
N = order of filter

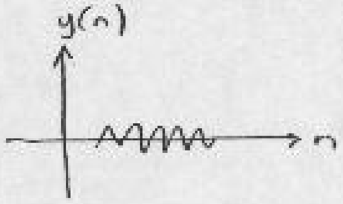
```
a= [...];
b= [...];
x= [a, b];
function    xq = quantx (x, n)          % n= # bits
            mx = max (abs(x));
            xn = x(mx);                 % x normalized
                                          coeff. (-1→1)
            delta = 2^(1-n);
            l1 = 1-delta;
            l2 = -1;
            xq = delta * round (xn/delta);
            lx = length (x);
            for i=1: lx
                if xq(i) > l1
                     xq(i) = l1;
                end
                if xq(i) < l2
                     xq(i) = l2;
                end
            end
            ax = xq (1: N+1);
            bx = xq (N+2 : end);
```

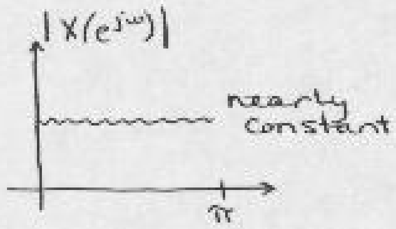
w = ...

```
w = ...
freqz( ...
roots (aq)                              % stability
if (any (abs (roots (aq))) > 1 )     % Needs to be
                                     % within unit
                                         circle
   disp ('Unstable Filter')
end
```

Gaussian Noise:





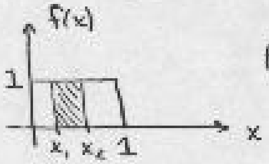


$$|H(e^{j\omega})| = \frac{|Y(e^{j\omega})|}{|X(e^{j\omega})|}$$

Creating Noise:

Random number generators

1) Uniform Random Number Generator
(#'s btwn 0 and 1)



Probability
Density
Function

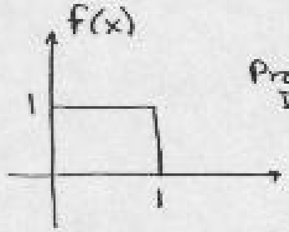
$$p(x_1 \le x \le x_2) = \int_{x_1}^{x_2} f(x)\,dx$$

$$p(0 \le x \le 1) = \text{area under curve} = 1$$

Average Value $= \bar{x} = E(x) = \int_0^1 x\, f(x)\, dx = 0.5$
$= \text{mean}$

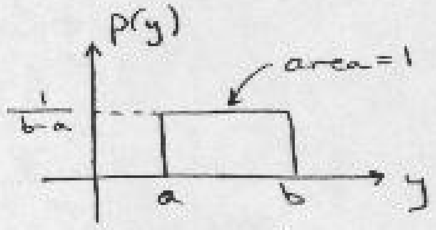
Standard Deviation $= \sigma$

$$\sigma^2 = E\left((x-\bar{x})^2\right) = \int_0^1 (x-0.5)^2\, dx$$



Prob. Density Function

$\xrightarrow{\text{Transformation}}$

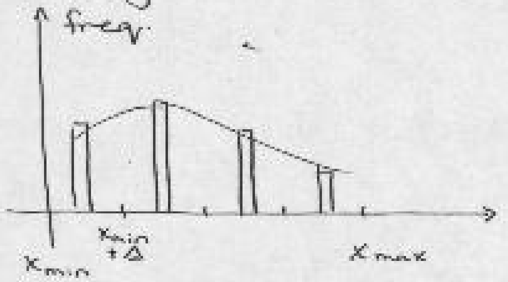


$y = (b-a)x + a$

Matlab Code:

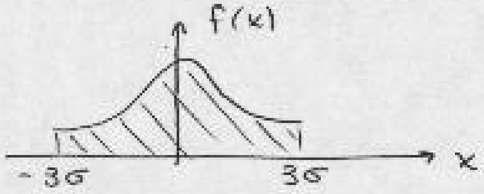
```
x= rand(n , 1)        % vector
y= (b-a)*x + a;
```

% Histogram



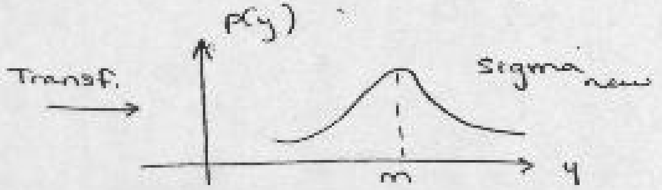
hist (x, nbins)
↳ default = 10

% normal (Guassian) Distributions



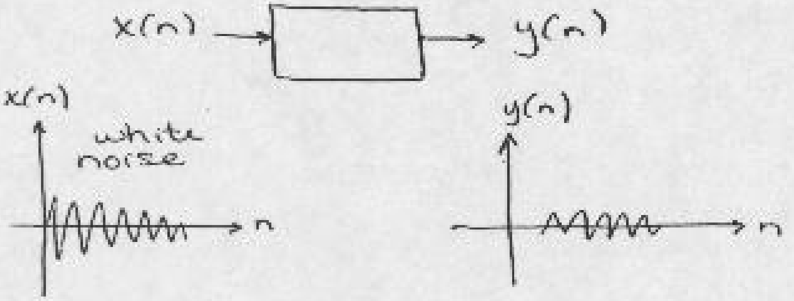
average = 0
$\sigma = 1$

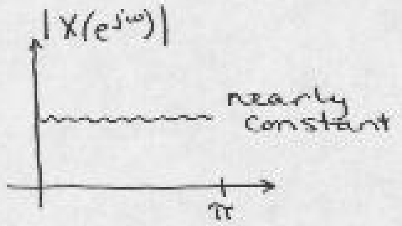
```
x = randn(m , 1)       % vector
mean(x) ≈ 0
std(x) ≈ 1
```



```
w= ...
freqz (...
roots (aq)                              % stability
if (any(abs(roots(aq))) >1)             % needs to be
                                        % within unit
                                        %        circle
   disp ('Unstable Filter')
end
```

Gaussian Noise:





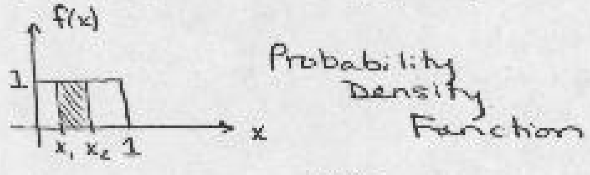
$$|H(e^{j\omega})| = \frac{|Y(e^{j\omega})|}{|X(e^{j\omega})|}$$

Creating Noise:

Random number generators

1) Uniform Random Number Generator

(#'s btwn 0 and 1)



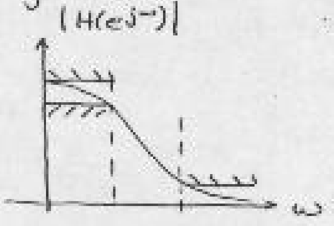
$$p(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} f(x)\,dx$$

$$p(0 \leq x \leq 1) = \text{area under curve} = 1$$

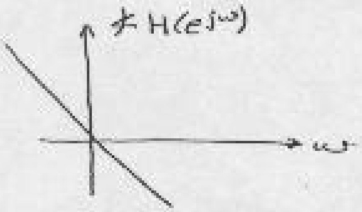
```
y= sigmanew * x + mnew        % Transformation

x= randn (10000, 1);
y= filter (b, a, x);
ω= [0: pi/1024 : pi];
Xspec = freqz (x, 1, ω);        % Plot is flat → Gaussian
Yspec = freqz (y, 1, ω);
|H| = Yspec ./ Xspec;
```

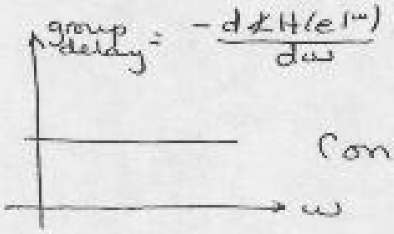
Design of FIR Filters



magnitude IIR



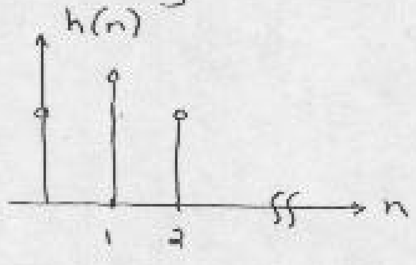
Linear Phase



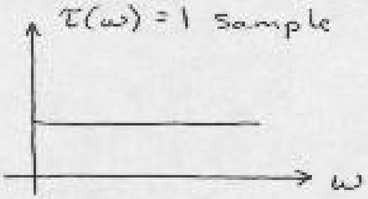
Constant Group Delay

Obtaining Linear Phase

①



Symmetrical about n=1

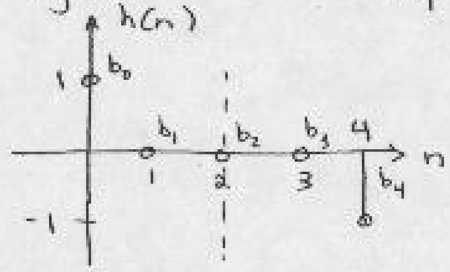


② Anti-symmetrical impulse response

ex: h(n)

② Anti-symmetrical impulse response

ex:



$h(4) = -h(0)$

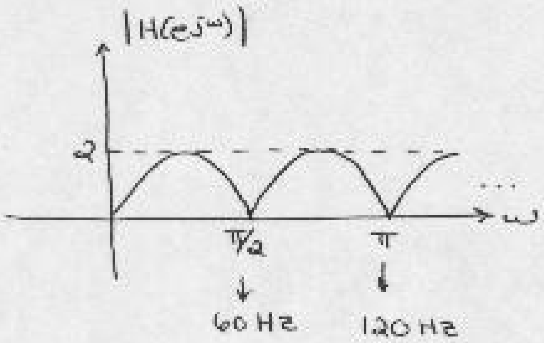
$h(n) = \delta(n) - \delta(n-4)$

Diff. Equ: $y(n) = x(n) - x(n-4)$

$y(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \cdots$

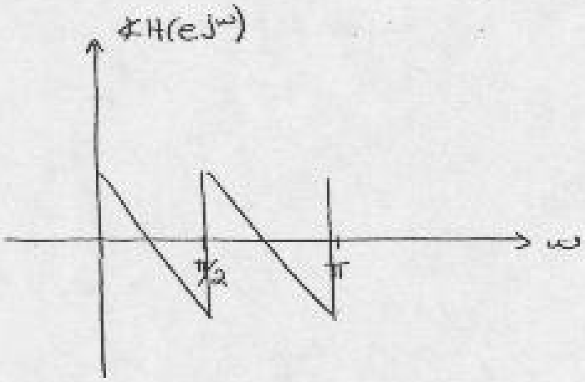
$$H(e^{j\omega}) = 1 - e^{-j4\omega} = e^{-j2\omega}\{e^{j2\omega} - e^{-j2\omega}\}$$
$$= 2je^{-j2\omega} \sin 2\omega$$

$$|H(e^{j\omega})| = 2|\sin 2\omega|$$



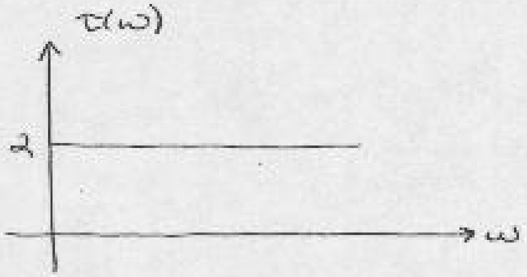
Comb Filter

$$\measuredangle H(e^{j\omega}) = \frac{\pi}{2} + (-2\omega) + \measuredangle \sin 2\omega$$
$$= \frac{\pi}{2} - 2\omega + \begin{cases} 0 & 0 \le \omega \le \pi/2 \\ \pi & \omega > \pi/2 \end{cases}$$

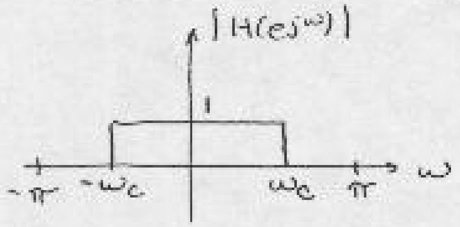


Generalized Linear

Group Delay: $\tau(\omega) = -\dfrac{d\measuredangle H(e^{j\omega})}{d\omega} = 2$ samples

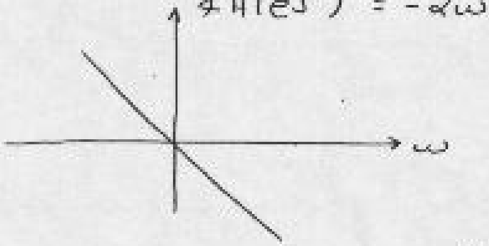


Consider the ideal LPF



One Period

$\measuredangle H(e^{j\omega}) = -\alpha\omega$ (α is assumed to be an integer)



$$H(e^{j\omega}) = |H(e^{j\omega})|\, e^{j\measuredangle H(e^{j\omega})}$$
$$= |H(e^{j\omega})|\, e^{-j\alpha\omega} = \begin{cases} 1e^{-j\alpha\omega}, & -\omega_c \le \omega \le \omega_c \\ 0, & \text{o.w.} \end{cases}$$

$\Rightarrow$ For one period

$$h_I(n) = FT^{-1}\{H(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} H(e^{j\omega})e^{j\omega n}\,d\omega$$
$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\alpha\omega}e^{jn\omega}\,d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\alpha)}\,d\omega$$

$n=\alpha$:

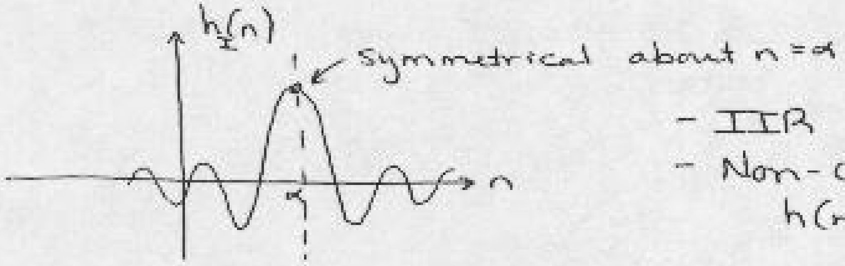
$$h_I(\alpha) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi} \qquad \frac{1}{2\pi}\left[\omega_c-(-\omega_c)\right] = \frac{\omega_c}{2\pi} + \frac{\omega_c}{2\pi} = \frac{\omega_c}{\pi}$$

$n \neq \alpha$:

$n \neq \alpha$:

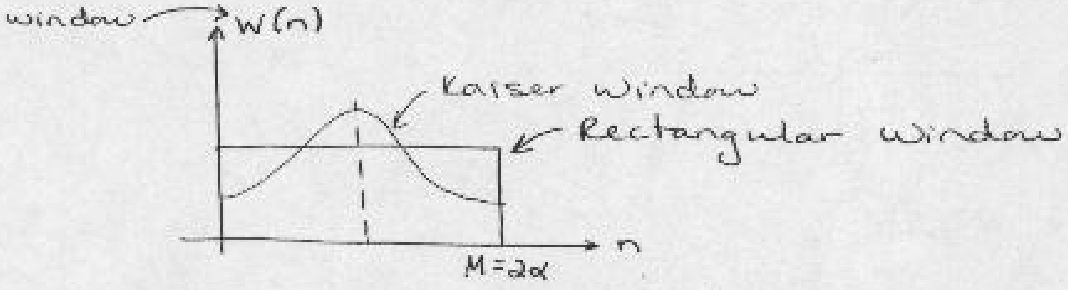
$$h_I(n) = \frac{1}{2\pi} \frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)} \Bigg|_{-\omega_c}^{\omega_c} = \frac{\sin \omega_c (n-\alpha)}{\pi(n-\alpha)} \quad , \quad n \neq \alpha$$

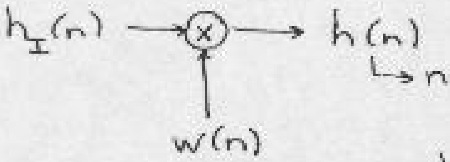
$$h_I(n) = \begin{cases} \dfrac{\sin \omega_c (n-\alpha)}{\pi(n-\alpha)} & , \quad n \neq \alpha \\ \dfrac{\omega_c}{\pi} & , \quad n = \alpha \end{cases}$$



- IIR
- Non-causal
  $h(n) \neq 0 \;\forall\; n < 0$

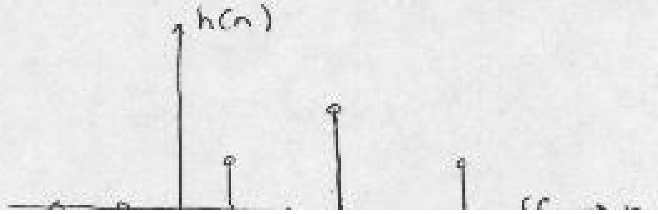
<u>In order to turn filter into FIR and causal:</u>

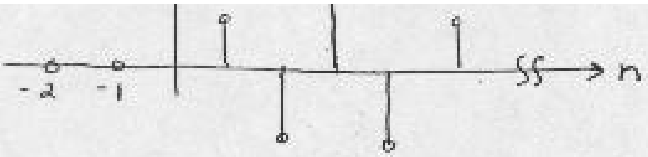


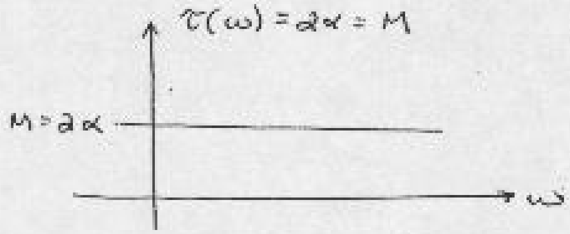


$\hookrightarrow n = 0, 1, 2, \dots, \underbrace{2\alpha}_{M}$

$h(n) = 0$, o.w.







Using windows to design FIR filters:

$$h(n) = h_I(n)\,w(n)$$

$$\text{FT}\downarrow \qquad \downarrow \text{FT}$$

$$H(e^{j\omega}) = \text{FT}\{h_I(n)\,w(n)\} = \underbrace{H_I(e^{j\omega}) * W(e^{j\omega})}$$

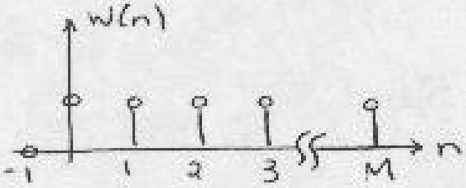
↳ Convolution (freq. domain)

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_I(e^{j\theta})\, W(e^{j(\omega-\theta)})\, d\theta$$

(θ: dummy Variable)

ex: Consider the rectangular window

$w(n) = 1 \quad , \qquad 0 \le n \le M$



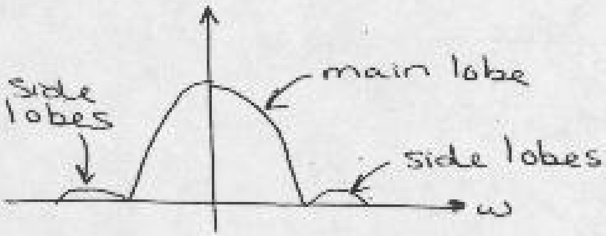
$$W(e^{j\omega}) = \text{FT}\{w(n)\} = \sum_{n=0}^{M} w(n)\, e^{-j\omega n}$$

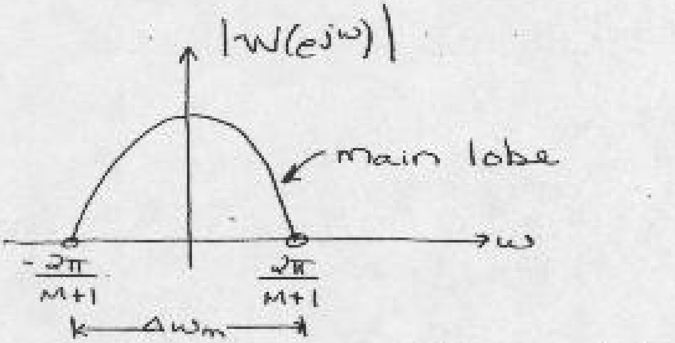
$$= \sum_{n=0}^{M} e^{-j\omega n} = \frac{1 - e^{-j\omega(M+1)}}{1 - e^{-j\omega}}$$

$$= \frac{e^{-j\omega\frac{(M+1)}{2}}}{e^{-j\omega/2}} \cdot \frac{\sin(\omega(M+1)/2)}{\sin(\omega/2)}$$

$$= e^{-j\omega M/2} \frac{\sin\left(\frac{\omega(M+1)}{2}\right)}{\sin(\omega/2)}$$

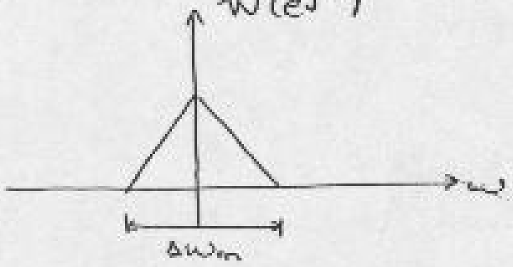
$$|W(e^{j\omega})| = \left| \frac{\sin\left(\frac{\omega(M+1)}{2}\right)}{\sin(\omega/2)} \right|$$





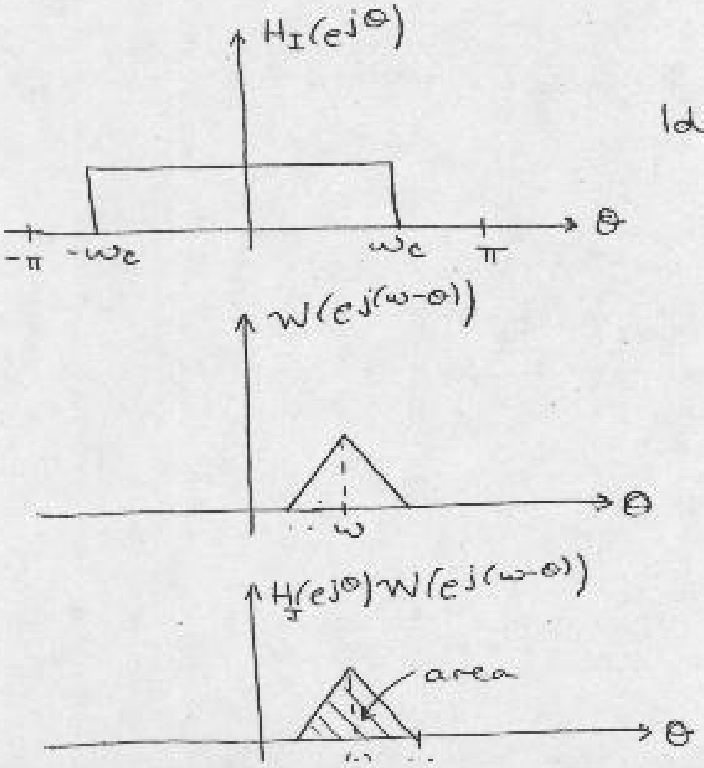
$\Delta\omega_m \equiv$ main lobe width

Assume that:



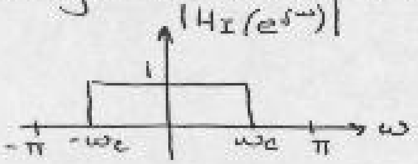
$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underset{\text{ideal}}{\underset{\downarrow}{H_I(e^{j\theta})}}\; \underset{\text{window}}{\underset{\downarrow}{W(e^{j(\omega-\theta)})}}\, d\theta$$

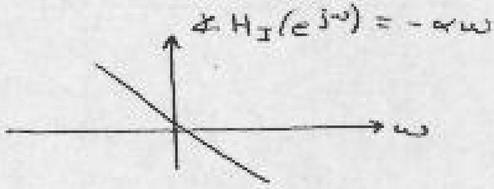
ideal LPF   window



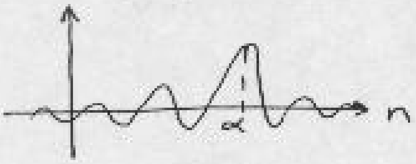
Ideal LPF

Design of FIR Filters

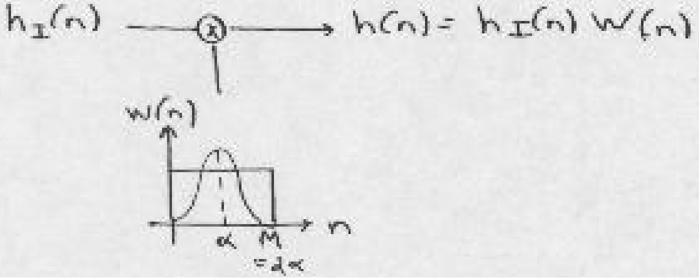




$$h_I(n) = FT^{-1}\{H(e^{j\omega})\}$$



- IIR
- Non-causal

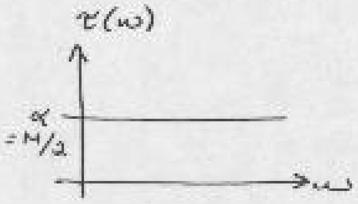


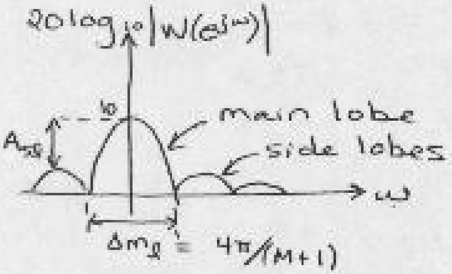
$h(n) = h_I(n)\,W(n)$

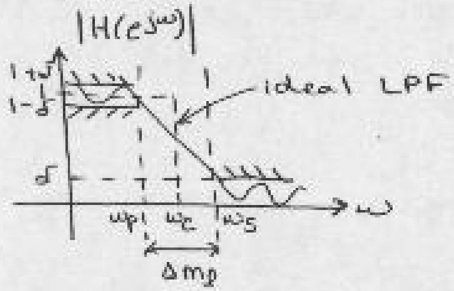
$n = 0, 1, 2, \cdots, M$

$M = 2\alpha$

$h(n) = 0$ outside this range







$$H(e^{j\omega}) = H_I(e^{j\omega}) * W(e^{j\omega})$$

↳ freq. conv.

To decrease lobe width,
increase # of samples, M

Design of FIR filters using the Kaiser Window method

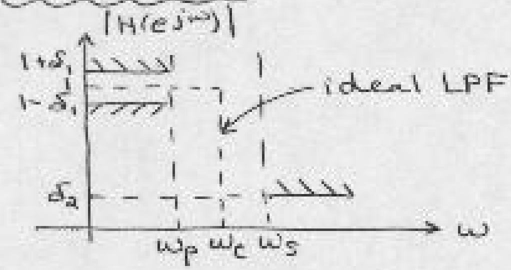
The Kaiser window has two parameters:

1) Number of samples (M+1)
2) β, parameter to control the side-lobes

$$W_k(n) = \begin{cases} \dfrac{I_0\left(\sqrt{\beta\left(1-\left(\frac{n-(M/2)}{M/2}\right)^2\right)}\right)}{I_0(\beta)} , & n = 0, 1, 2, \ldots, M \\ 0 , & \text{o.w.} \end{cases}$$

$I_0(\beta)$ = Zeroth-order Bessel function of the first kind

For a LPF:



$$\omega_c = \frac{\omega_p + \omega_s}{2} \quad ①$$

If $\delta_1 \neq \delta_2 \Rightarrow$ take $\delta = \min\{\delta_1, \delta_2\}$

$$A = -20 \log_{10} \delta \quad ②$$

↳ stop band attenuation

$$\beta = \begin{cases} 0.1102(A-8.7), & A > 50 \text{ dB} \\ 0.5842(A-21)^{0.4} + 0.0788(A-21), & 21 \le A \le 50 \text{ dB} \\ 0, & A < 21 \text{ dB} \end{cases} \quad ③$$

↳ Rect. window

$\Delta\omega$ (transition band) $= \omega_s - \omega_p = \Delta ml$

$$M = \frac{A-8}{2.285\,\Delta\omega}$$

(within ±2 samples)
(round up to closest even integer)

Example: $\delta_1 = 0.01$, $\delta_2 = 0.001$, $\omega_p = 0.4\pi$ rad, $\omega_s = 0.6\pi$ ra
Design LPF w/ given info.

$\delta_1 \neq \delta_2 \Rightarrow \delta = \min\{\delta_1, \delta_2\} = 0.001$

$A = -20\log_{10}\delta = -20\log_{10}(0.001) = 60$ dB

$\beta = 0.1102(A-8.7)$ because $A > 50$ dB

$= 0.1102(60-8.7) = 5.653$

$$M = \frac{A-8}{2.285\Delta\omega} = \frac{60-8}{2.285\,\Delta\omega} = \frac{60-8}{2.285(\omega_s-\omega_p)}$$

$$= \frac{52}{2.285(0.2\pi)} = 37$$

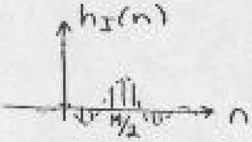
$\Rightarrow$ Take $M = 38$ (even) $\Rightarrow \tau = 38/2 = 19$ samples

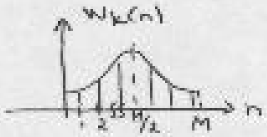
$\Rightarrow$ Take $M = 38$ (even) $\Rightarrow \tau = 38/2 = 19$ samples

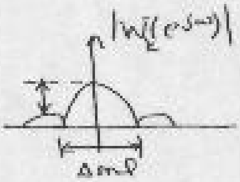
$$\omega_c = \frac{\omega_p + \omega_s}{2} = 0.5\pi \text{ rad.}$$

$$h_I(n) = \begin{cases} \dfrac{\sin \omega_c (n - M/2)}{\pi (n - M/2)} , & n \neq M/2 \\ \dfrac{\omega_c}{\pi} , & n = M/2 \end{cases}$$

$$h(n) = h_I(n)\, w_K(n)$$

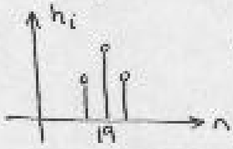


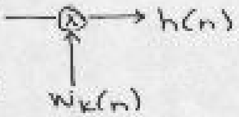




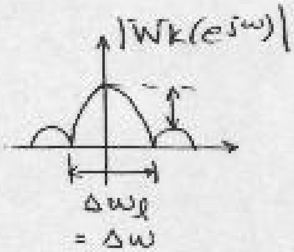
matlab Code:

```
M = 38;
beta = 5.653;
wc = 0.5 * pi;
n = [0:M]';
hi = sin(wc*(n-M/2))./(pi*(n-M/2));
ind = find (n == M/2);
hi(ind) = wc/pi;
stem (n, hi)
```



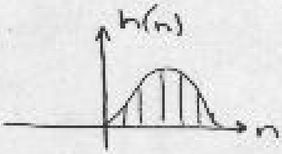


```
wk = kaiser(M+1, beta);
stem (n, wk);
w = [-pi : pi/512 : pi]';
WK = freqz (wk, 1, w);
magWK = abs (WK);
plot (w, magWK);
```

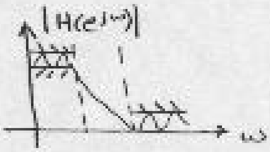


```
h = hi .* wk;
```

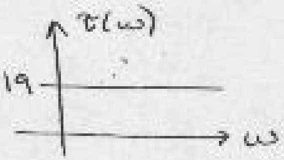
```
h = hi .* wk;
stem(n, h);                      % make sure meets
                                   requirements
```



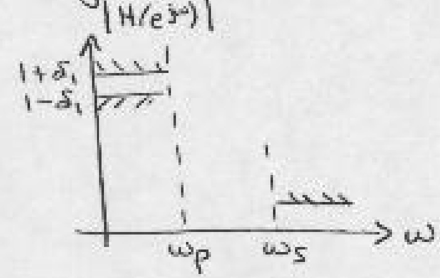
```
H = freqz(h, 1, w);
magH = abs(H);
plot(w, magH);                   % make sure meets
                                   requirements
```



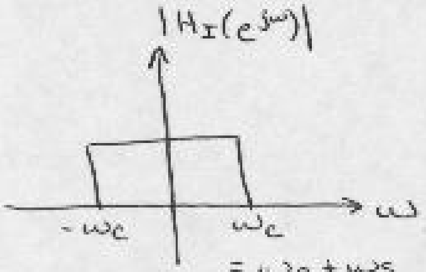
```
gdH = grpdelay(h, 1, w);
```



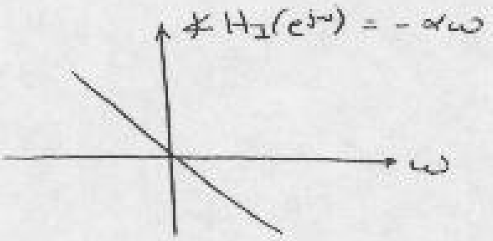
Design of FIR filters



Linear Phase ≡ Constant Group Delay



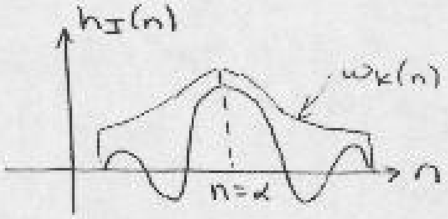
$$= \frac{\omega_p + \omega_s}{2} \quad ①$$

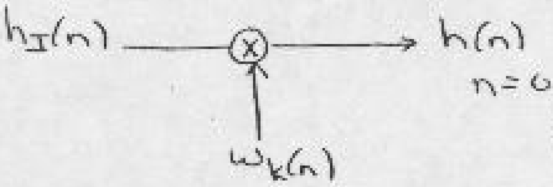


$$\delta = \min\{\delta_1, \delta_2\} \quad ②$$

$$h_I(n) = FT^{-1}\{H_I(e^{j\omega})\}$$

↑ Ideal LPF Impulse Response





$n = 0, 1, 2, \cdots, M$

$M = 2\alpha$

$$A = -20 \log_{10} \delta$$

$$\beta = \begin{cases} \text{—} \\ \text{—} \end{cases}$$

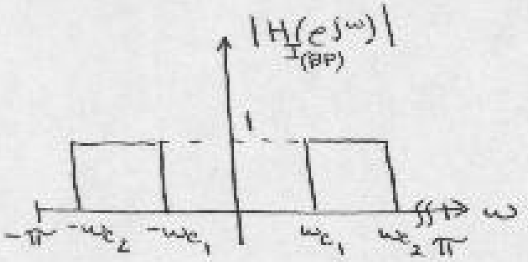
$$\beta = \begin{cases} - \\ - \end{cases}$$

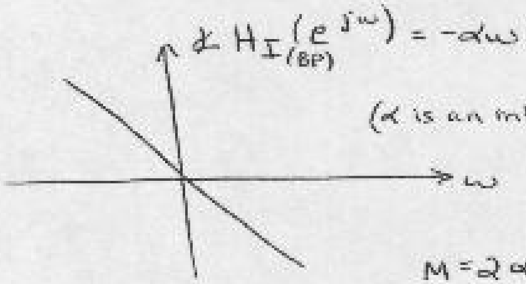
$$M = \frac{A-8}{2.285\Delta\omega} \quad \text{(within } \pm 2\text{)}$$

$\hookrightarrow (\omega_s - \omega_p)$

Extending the window method to other types of filters (high pass, band pass, notch)

Consider the ideal BPF:





(α is an integer)

$M = 2\alpha$

even integer

$$H_{I(BP)}(e^{j\omega}) = \begin{cases} e^{-j\omega M/2} & , \quad \omega_{c_1} \le |\omega| \le \omega_{c_2} \\ 0 & , \quad \text{o.w.} \end{cases}$$

$$h_{I(BP)}(n) = FT^{-1}\{H_{I(BP)}(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{I(BP)}(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_{c_2}}^{-\omega_{c_1}} e^{j\omega(n-M/2)}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c_1}}^{\omega_{c_2}} e^{j\omega(n-M/2)}\, d\omega$$

$$h_{I(BP)}(n) = \begin{cases} \frac{1}{\pi(n-M/2)}\{\sin \omega_{c_2}(n-M/2) - \sin \omega_{c_1}(n-M/2) & , \quad n \neq M/2 \\ \frac{\omega_{c_2}-\omega_{c_1}}{\pi} & , \quad n = M/2 \end{cases}$$

Special Cases:

① $\omega_{c_1} = 0 \Rightarrow$ LPF

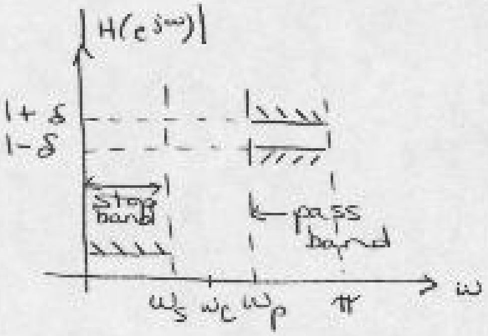
① $\omega_{c_1} = 0 \Rightarrow$ LPF
② $\omega_{c_2} = \pi \Rightarrow$ HPF

$$\therefore \quad h_{I(HP)}(n) = \begin{cases} \dfrac{-\sin \omega_{c_1}(n - M/2)}{\pi(n - M/2)} , & n \neq M/2 \\ 1 - \omega_{c_1}/\pi , & n = M/2 \end{cases}$$

Example:

Design a HPF with linear phase to meet the following specs:

$\omega_s = 0.35\pi \quad \omega_p = 0.5\pi \quad \delta = 0.021$



$A = -20 \log \delta = 33.55$ dB

$\beta = 0.5842\,(33.55 - 21)^{0.4} + 0.07886\,(33.55 - 21) = 2.6$

$\Delta\omega = \omega_p - \omega_s = 0.15\pi$

$M = \dfrac{A - 8}{2.285\,\Delta\omega} = 24$ (order)

$\Rightarrow 25$ samples

$\omega_c = \dfrac{\omega_p + \omega_s}{2}$

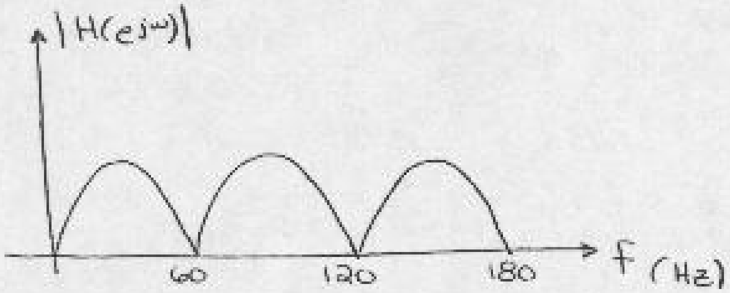
Matlab:

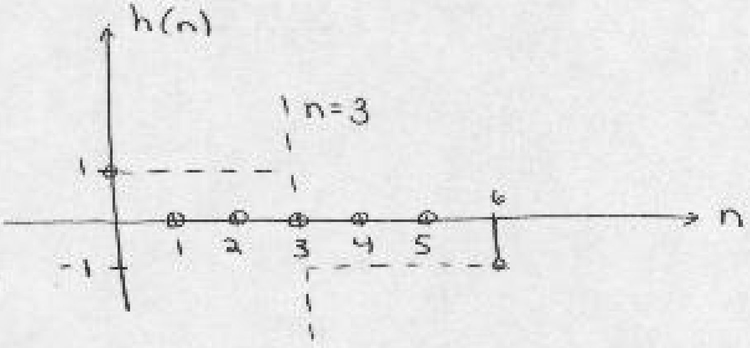
```
beta = ... ;
M = ... ;
wc = ... ;
n= [0:M]
hihp = -sin(wc * (n- M/2))./(pi * (n-M/2));
                                    % h:ideal high pass
ind = find(n==M/2);
hihp(ind)= (1- wc/pi);
% then plot response
% Need to truncate using Kaiser window
kw= kaiser(M+1, beta);        % #samples = M+1
% Plot Kaiser window
h= hihp.* kw;
```

$$\left\{\begin{array}{l} \text{If } h = [0.1 \;\; -0.2 \;\; 5 \; \cdots] \\ \text{then } \; y(n) = \underset{\substack{\downarrow \\ h(1)}}{b_0} x(n) + \underset{\substack{\downarrow \\ h(2)}}{b_1} x(n-1) + \cdots \\ h \text{ is equivalent to } b \text{ vector} \end{array}\right.$$

```
% Find freq. resp.
w= [0: pi/1024: pi];
H= freqz(h,1,w);
% Group Delay must be constant and equal
   to a value of M/2
```

<u>Other FIR filters</u>





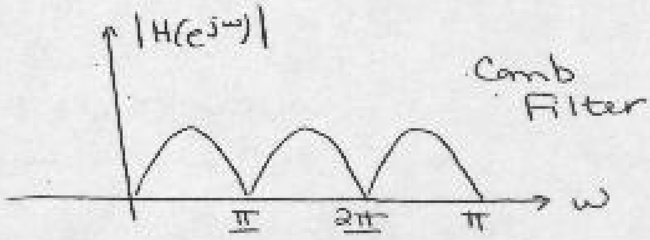
Anti-Symmetrical impulse response

$$\underset{\downarrow FT}{h(n)} = \underset{\downarrow FT}{\delta(n)} - \underset{\downarrow FT}{\delta(n-6)}$$

$$H(e^{j\omega}) = 1 - e^{-j6\omega} = e^{-j3\omega}\{e^{j3\omega} - e^{-j3\omega}\}$$

$$= 2je^{-j3\omega} \sin 3\omega$$

$$|H(e^{j\omega})| = 2|\sin 3\omega|$$



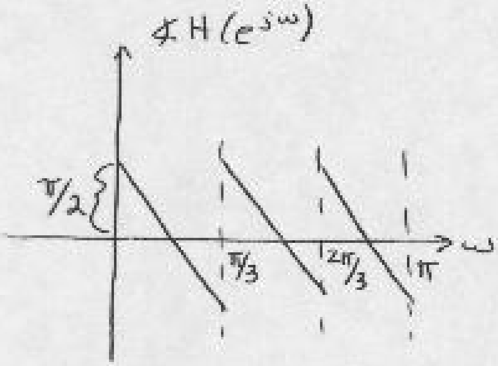
Comb Filter

$\omega = \pi/3 \longrightarrow 60 \text{ Hz}$

$\omega = \pi \longrightarrow 180 \text{ Hz}$

$\therefore f_{samp} = 360 \text{ Hz}$

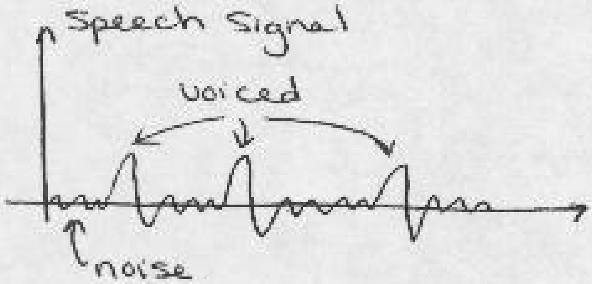
$$\measuredangle H(e^{j\omega}) = \frac{\pi}{2} - 3\omega - \begin{cases} 0 & , \quad 0 \le \omega < \pi/3 \\ \pi & , \quad \pi/3 \le \omega < 2\pi/3 \\ 0 & , \quad 2\pi/3 \le \omega < \pi \end{cases}$$



$$\tau(\omega) = \frac{-d \measuredangle H(e^{j\omega})}{d\omega} = 3 \text{ samples.}$$

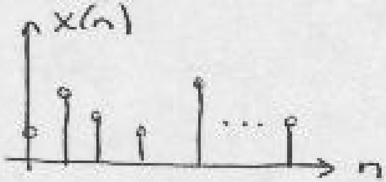
Time-domain methods of Signal Analysis

mini-project 2



1) Short-time energy
2) Short-time magnitude
3) Short-time zero-crossing

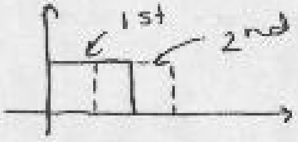
Energy



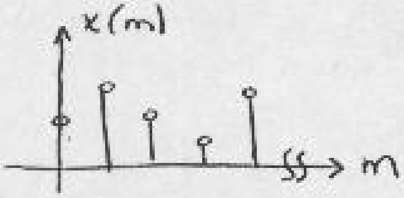
$$\mathcal{E} = \sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$$

Energy of a Sequence

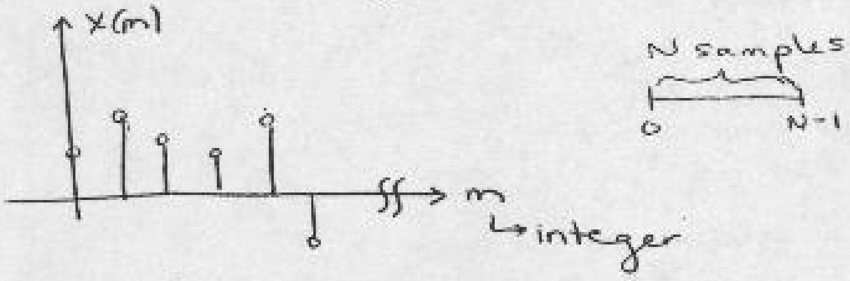
Use windows, continue to slide window down axis (w/ overlapping) to not miss events



Short-time energy

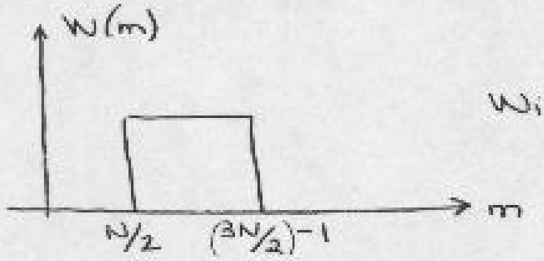


Short-time Energy



$$\mathcal{E}_1 = \sum_{m=-\infty}^{\infty} | x(m) w(m) |^2$$

$$\mathcal{E}_2 = \sum_{m=-\infty}^{\infty} | x(m) w(m - N/2) |^2$$

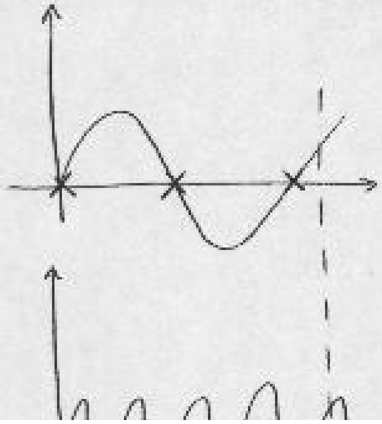


Window (Usually 10 or 20 msec)

Short-time Magnitude

$$M_n = \sum_{m=-\infty}^{\infty} | x(m) | w(n-m)$$

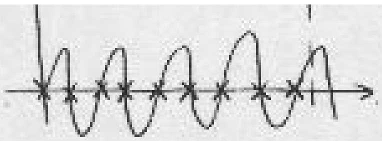
Voice vs. Unvoiced Speech



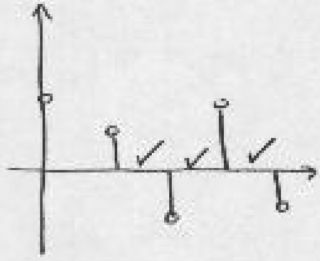
freq f_a (voiced)

$f_b >> f_a$

freq f_b (unvoiced)



You have a zero crossing when signs change



3 zero crossings

Use Hamming window: